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COMPACTIFICATIONS OF CLUSTER VARIETIES ASSOCIATED TO ROOT SYSTEMS

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COMPACTIFICATIONS OF CLUSTER VARIETIES ASSOCIATED TO ROOT SYSTEMS

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ABSTRACT

CLUSTER VARIETIES CORRESPONDING TO

THE ROOT SYSTEMS

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In this thesis we identify certain cluster varieties with the complement of a union of closures of hypertori in a toric variety. We prove the existence of a compactification Z of the Fock–Goncharov $\mathcal{X}$-cluster variety for a root system Φ satisfying some conditions, and study the geometric properties of Z. We give a relation of the cluster variety to the toric variety for the fan of Weyl chambers and use a modular interpretation of $X(A_n)$ to give another compactification of the $\mathcal{X}$-cluster variety for the root system A_n.

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CHAPTER 1

INTRODUCTION

Cluster algebras were introduced by S. Fomin and A. Zelevinsky in 2002 in a series of papers [FZ02], [FZ03]. One of their goals was to develop an algebraic framework for understanding Lusztig's dual canonical bases and total positivity. In [FG2], Fock and Goncharov formalized the framework for a geometric approach to cluster algebras. They conjectured the existence of canonical bases of global regular functions on cluster varieties.

Gross, Hacking, and Keel interpreted cluster varieties from the viewpoint of birational geometry, realizing that cluster varieties are an ideal candidate for beginning to generalize their log Calabi-Yau surface constructions [GHKI] to higher dimensions. They gave a geometric interpretation of cluster varieties in terms of blowups of toric varieties in [GHKII] and gave an elementary geometric proof of the Laurent phenomenon for cluster algebras of geometric type. Together with Kontsevich in [GHKK], they used their techniques to construct the bases conjectured by Fock and Goncharov.

Here is a brief introduction to my thesis.

In Chapter 2, for each root system Φ, we identify the $\mathcal{X}$-cluster variety associated to Φ with the complement of a union of closures of hypertori H_i and some toric boundary divisors in a toric variety X up to codimension two. This uses the geometric interpretation of cluster varieties in terms of blowups of toric varieties in [GHKII].

Theorem 1.1 *(Theorem 2.8) Let Φ be a simply laced root system. Let X be the toric variety corresponding to the root system Φ, which is the blowup of some loci in $(\mathbb{P}^1)^n$ in some order. Let $U = X \setminus D$, where $D = \sum_{i=1}^n H_i + \sum_{i=1}^n B_i$, H_i is the closure of a hypertorus, and B_i is a toric boundary divisor.(See Theorem 2.8 for details.) Then U is a log Calabi-Yau variety, and U is isomorphic to the $\mathcal{X}$ cluster variety for the root system Φ up to codimension two.*

We generalize Theorem 1.1 to any finite type root systems in Theorem 2.11.

In Theorem 1.1 we treat the cluster variety as toric variety with a non-toric boundary divisor removed. We also give a toric degeneration from a $\mathcal{X}$-cluster variety U to a torus.

In Chapter 3, we consider the sequence of birational modifications, which are isomorphisms in codimension one, starting with a blowup X of $(\mathbb{P}^1)^n$, through a toric variety Y such that $-K_Y$ is nef, and ending with a toric variety Z such that $-K_Z$ is ample.

$$\begin{array}{ccccc} X & \dashrightarrow & Y & \longrightarrow & Z \\ \downarrow & & & & \\ (\mathbb{P}^1)^n & & & & \end{array}$$

The sequence of birational modifications $X \dashrightarrow Y$ has a combinatorial description in terms of the secondary fan associated to the set of rays in the fan of X.

We can find it in the book Toric Varieties [CLS11]. It is a special case of the procedure 15.5.5 on p. 776–798 and the explicit algorithm in the proof of Proposition 15.5.6 on p. 777–799. We apply this algorithm with $D = -K_X$ to get the sequence of birational modifications from X to Y. Then $-K_Y$ is nef and defines a morphism $Y \longrightarrow Z$ which is birational and an isomorphism in codimension one, and is determined by a canonical coarsening of the fan of Y. Namely, the fan Σ_Z consists of the cones over the faces of the convex hull of the primitive generators of the rays in the fan Σ_Y of Y. In our case, $Y \longrightarrow Z$ is an isomorphism in codimension one.

The following theorem gives a compactification of the $\mathcal{X}$-cluster variety for a root system up to codimension 2.

Theorem 1.2 *(Theorem 3.10, Theorem 3.22)*

Let U be a $\mathcal{X}$-cluster variety of finite type for the simply laced root system Φ. Then there exists a compactification $U = Z \setminus D_Z$ up to codimension two such that the following conditions hold:

1. *Z is a toric variety with terminal singularities and $-K_Z$ is ample.*
2. *$D_Z = H_{1,Z} \cup ... \cup H_{n,Z} \bigcup B_{1,Z} \cup ... \cup B_{n,Z}$, where $H_{i,Z}$ are the corresponding hypertori $H_{i,Z} = \overline{(\mathcal{X}^{e_i} = 1)}$ in Z, and $B_{i,Z}$ are the toric boundary divisors of Z corresponding to the rays $\mathbb{R}_{\geq 0} \cdot (-f_i)$, $i = 1, ..., n$.*
3. *$K_Z + D_Z = 0$. In particular, D_Z is ample and U_Z is affine.*
4. *(Z, D_Z) is log canonical.*

We then analyze the properties of the toric variety Z by analyzing the corresponding polytope P. By showing that P is a terminal polytope, we prove that Z has terminal singularities.

Using the program [PALP], we checked that P is a reflexive polytope for A_n, D_n, when $n \leq 19$, and E_6, E_7, E_8 cases. Then we have the following conjecture.

Conjecture 1.3 *(Conjecture 3.24) P is a reflexive polytope for simply laced root systems, and so Z is a Gorenstein Fano toric variety for simply laced root system.*

Above is the main part of this thesis.

In Chapter 4, we relate the well-known toric variety $X(\Phi)$ associated with the root system with the cluster variety for the root system defined in Theorem 1.1. We see that X is a toric open set in $X(\Phi)$ up to codimension two.

We also try to use a big polytope $\tilde{P}$, which is the polytope corresponding to $X(\Phi)$, to prove Conjecture 1.3, but it does not work. We give an alternative approach to prove P is a terminal polytope when $\Phi = A_n$.

In Chapter 5, we use a modular interpretation of $X(A_n)$ to find morphisms of some weighted moduli spaces, and this gives another compactification of the $\mathcal{X}$-cluster variety for the root system A_n, which is a Gorenstein terminal toric Fano variety.

CHAPTER 2

THE CLUSTER VARIETY IS AN OPEN SUBSET OF A TORIC VARIETY

2.1 Introduction to cluster algebra and cluster variety

Cluster algebras are a class of commutative rings introduced by S. Fomin and A. Zelevinsky in 2002 in a series of papers [FZ02], [FZ03].

Roughly speaking, a cluster algebra $\mathcal{A}$ is a certain subalgebra of $k(x_1, x_2, ..., x_n)$, the field of rational functions in the variables $\{x_1, x_2, ..., x_n\}$ over the field k. Generators are constructed by a series of exchange relations which induce all relations satisfied by the generators.

A seed for $\mathcal{A}$ is a initial cluster $\{x_1, x_2, ..., x_n\}$, together with an $n \times n$ skew-symmetrizable integral matrix B. For any seed, we can mutate it in n directions to get another n seeds. The mutation is defined by the matrix B, see [FZ02] for more details. Starting with the initial cluster, and doing all possible sequences of mutations, will produce the set of all cluster variables. Now the cluster algebra $\mathcal{A}$ is a subring of $k(x_1, x_2, ..., x_n)$ generated by all cluster variables.

If a cluster algebra has only a finite number of seeds, then we say it is a finite type cluster algebra.

Fomin and Zelevinsky showed that the cluster algebras of finite type can be classified in terms of the Cartan-Killing classification of complex simple Lie algebras.

Theorem 2.1 ([FZ03], Theorem 1.8) *There is a canonical bijection between the Cartan matrices of finite type and the strong isomorphism classes of cluster algebras of finite type.*

By the Cartan-Killing classification, we can classify irreducible root systems by classifying the corresponding Dynkin diagrams. In particular, cluster algebras of finite type correspond to one of the infinite series A_n, B_n, C_n, D_n, $n = 1, 2, ...$, or to one of the exceptional types E_6, E_7, E_8, F_4, G_2; we say that the cluster algebra $\mathcal{A}$ is of type X, where X is one of the root systems.

Let Φ be a root system. Fomin and Zelevinsky also showed that the cluster variables of $\mathcal{A}$ are naturally parameterized by the set $\Phi_{\geq 1}$ of almost positive roots.

Theorem 2.2 ([FZ03], Theorem 1.9) *There is a canonicial bijection $\alpha \to x[\alpha]$ between the almost positive roots in Φ and the cluster variables in $\mathcal{A}$.*

A. Zelevinsky's student J. Scott studied the cluster structure of the homogeneous coordinate ring of the Grassmannian $G(k, n)$ by using the Postnikov diagrams in [JS]. He gave a classification of Grassmannians of finite type: $G(2, n), G(3, 6), G(3, 7)$, and $G(3, 8)$, which correspond to the root systems A_n, D_4, E_6, and E_8.

Fock and Goncharov [FG2] constructed two types of cluster varieties by gluing tori using the data related to the cluster algebras of Fomin and Zelevinsky. They are called cluster ensembles $(\mathcal{A}, \mathcal{X})$.

Following [FG2], we have some fixed data:

Let $N = \mathbb{Z}^n = N_{\mathcal{A}} = M_{\mathcal{X}}$ be a lattice. Let $M = \mathrm{Hom}(N, \mathbb{Z})$ be the dual lattice of N.

Let $\{\cdot, \cdot\} : N_{\mathcal{A}} \times N_{\mathcal{A}} \longrightarrow \mathbb{Z}$ be a skew-symmetric bilinear form on $N_{\mathcal{A}}$.

Given these fixed data, a seed data $\mathbf{s} := (e_i | i = 1, 2, ..., n)$ for this fixed data is a collection of elements of N such that $(e_i | i = 1, 2, ..., n)$ is a basis of N.

Given a seed $\mathbf{s}$, we obtain a dual basis f_i for M, denoted by $f_i = e_i^*$.

Let $v_i = \{e_i, \cdot\} \in M = N^* = M_{\mathcal{A}} = N_{\mathcal{X}}$, then $v_i = \sum b_{ij} f_j$, where $b_{ij} = \{e_i, e_j\}$ are determined by the bilinear form $\{\cdot, \cdot\}$, and $f_1, ..., f_n = e_1^*, ..., e_n^*$ are the dual basis of $N_{\mathcal{X}}$.

Given seed data $\mathbf{s}$, we can associate two tori $\mathcal{X}_{\mathbf{s}} = T_M = \mathrm{Spec}\, k[N]$ and $\mathcal{A}_{\mathbf{s}} = T_N = \mathrm{Spec}\, k[M]$.

An element $\alpha \in N$ gives us a character $\mathcal{X}^\alpha$ of the torus T_M. Its value on a homomorphism $x \in T$ is $x(\alpha)$. Let $X_i = \mathcal{X}^{e_i}$. Then we have the cluster $\mathcal{X}$-coordinates X_i, $i = 1, ..., n$, where $X_1, X_2, ..., X_n$ are a basis in the group of characters of the torus T_M. Let $A_i = \mathcal{X}^{f_i}$, we have the cluster $\mathcal{A}$-coordinates A_i, $i = 1, ..., n$, where $A_1, A_2, ..., A_n$ are a basis in the group of characters of the torus T_N. The coordinates X_i, A_i are called cluster variables.

Now let Φ be a root system and N the lattice generated by the roots, with basis $e_1, ..., e_n$, a basis of simple roots. For any $\alpha \in \Phi$, we have character $\mathcal{X}^\alpha : T \to \mathbb{C}^\times$. We can consider the hypertorus $H_\alpha = \overline{(\mathcal{X}^\alpha = 1)}$.

Now we can glue these $\mathcal{A}$ or $\mathcal{X}$ type tori along the birational maps defined by the mutation formulas to construct a cluster $\mathcal{A}$ variety or $\mathcal{X}$ variety, i.e., $\mathcal{A} = \bigcup_{\mathbf{s}} \mathcal{A}_{\mathbf{s}}$ and $\mathcal{X} = \bigcup_{\mathbf{s}} \mathcal{X}_{\mathbf{s}}$, where the gluing maps are defined by the mutation of seeds.

In [GHKII], they give a simple explanation of cluster varieties in terms of blowups of toric varieties. Each seed in the cluster algebra gives a description of the $\mathcal{A}$ or $\mathcal{X}$ cluster variety up to codimension two as a blowup of a toric variety.

The mutation of the seed corresponds to changing the blowup description by an elementary transformation of a $\mathbb{P}^1$-bundle. We will follow this description.

Cluster varieties are essentially the log Calabi-Yau varieties which are holomorphic symplectic and admit toric models from the view of [GHKII].

Definition 2.3 *[HK] Suppose X is a smooth projective variety, $D \subset X$ is a normal crossing divisor on X, if $K_X + D = 0$, then we say (X, D) is a log Calabi-Yau pair. A variety U is called* log Calabi-Yau *if it has a smooth projective compactification X with normal crossing boundary D such that $K_X + D = 0$.*

Definition 2.4 *[HK] A* toric model *of a log Calabi-Yau variety U is a log Calabi-Yau compactification (X, D) of U together with a birational morphism $f : (X, D) \longrightarrow (\bar{X}, \bar{D})$ such that $(\bar{X}, \bar{D})$ is a toric variety together with its toric boundary and f is a composition of blow ups.*

Definition 2.5 *[HK] A log Calabi-Yau variety U is a* cluster variety *if*

1) There is a non-degenerate holomorphic 2-form σ on U such that for some normal crossing compactification (X, D) we have $\sigma \in H^0(\Omega^2_X(logD))$.

2) U has a toric model.

We have a toric model for any cluster variety.

2.2 Cluster variety as an open subset of toric variety

In this thesis, we will focus on the $\mathcal{X}$-cluster variety. We give more details of the description of the $\mathcal{X}$ cluster variety up to codimension two as a blowup of a toric variety. This follows from [GHKII].

Let $\mathbf{s}$ be a seed in the cluster variety $\mathcal{X}$. We consider the fan

$$\Sigma_{\mathbf{s},\mathcal{X}} = \{0\} \cup \{\mathbb{R}_{\geq 0} \cdot (-v_i), i = 1, ..., n\}.$$

Let $TV_{\mathbf{s},\mathcal{X}}$ be the toric variety defined by the fan $\Sigma_{\mathbf{s},\mathcal{X}}$. Let D_i be the toric boundary divisor corresponding to the ray $\mathbb{R}_{\geq 0} \cdot (-v_i)$ in the fan $\Sigma_{\mathbf{s},\mathcal{X}}$. Define a closed subvariety

$$Z_i = Z_{\mathcal{X},i} := (1 + \mathcal{X}^{e_i} = 0) \cap D_i.$$

Let $\tilde{TV}_{\mathbf{s},\mathcal{X}}$ be the blowup of $TV_{\mathbf{s},\mathcal{X}}$ along the closed subvarieties Z_i, i.e., $\tilde{TV}_{\mathbf{s},\mathcal{X}} = Bl_{Z_1,...,Z_n} TV_{\mathbf{s},\mathcal{X}}$.

For the pair $(\tilde{TV}_{\mathbf{s},\mathcal{X}}, D')$, we define

$$U_{\mathbf{s},\mathcal{X}} := \tilde{TV}_{\mathbf{s},\mathcal{X}} \setminus D',$$

where $D' = \sum_{i=1}^{n} D'_i$, and D'_i is the strict transform of toric boundary divisors D_i.

For the $\mathcal{X}$ cluster variety, we can have $v_i = av_j$ for some i, j and $a > 0$, so then $D_i = D_j$. This means the two centers Z_i and Z_j may intersect, i.e., $Z_i \cap Z_j \neq \emptyset$, but we see that $\dim\ Z_i \cap Z_j < \dim\ Z_i$, since we have

$$Z_i \cap Z_j = (1 + \mathcal{X}^{e_i} = 0) \cap (1 + \mathcal{X}^{e_j} = 0) \cap D_i.$$

This means Z_i and Z_j only intersect in higher codimension.

We say a birational map $X \dashrightarrow Y$ is an isomorphism up to codimension two if there exist $Z \subset X$, $W \subset Y$ of codimension ≥ 2, such that

$$U = X \setminus Z \simeq V = Y \setminus W.$$

Lemma 2.6 *[GHKII] Let Φ be a root system and $\mathcal{D}$ be a oriented Dynkin diagram corresponding to Φ. Then for the seed $\mathbf{s}$ corresponding to $\mathcal{D}$, $U_{\mathbf{s},\mathcal{X}}$ is isomorphic to the cluster variety $\mathcal{X}$ up to codimension two.*

For a root system Φ, let $\Delta = \{\alpha_1, ...\alpha_n\}$ be a basis of simple roots of Φ. We have the Cartan matrix $A = (a_{ij})$, where $a_{ij} = (\alpha_i, \alpha_j)$ for $1 \leq i, j \leq n$. Then A is a symmetric matrix.

Let $B = (b_{ij})$ be a matrix which corresponds to a choice of orientation of the edges of the Dynkin diagram of Φ. Two vertices i, j are joined by an edge if $b_{ij} \neq 0$. If $b_{ij} > 0$, then the edge is oriented from i to j; if $b_{ij} < 0$, then the edge is oriented from j to i. We have $a_{ii} = 2$, $a_{ij} = -|b_{ij}|$ for $i \neq j$, so $(\alpha_i, \alpha_j) = -|\{e_i, e_j\}|$ for $i \neq j$.

Let us assume for the seed tori we consider, that every vertex is a "source" or a "sink", this means that for every i, we have $b_{ij} \geq 0, \forall j$, or $b_{ij} \leq 0, \forall j$. In terms of $B = (b_{ij})$, this means that each row has all entries ≥ 0 or all entries ≤ 0, and also means that each column has all entries ≥ 0 or all entries ≤ 0, since the matrix B is skew symmetric.

For simply laced root systems,

$$|b_{jk}| = \begin{cases} 1, & \text{if } jk \text{ is an edge} \\ 0, & \text{if } jk \text{ is not an edge} \end{cases}$$

Let $\rho_i = \mathbb{R}_{\geq 0} \cdot v_i$ for $i = 1, .., 3n$, where

$$v_i = f_i, \; v_{n+i} = f_i - \sum_{k=1, k\neq i}^{n} |b_{ik}| \cdot f_k, \; v_{2n+i} = -f_i, \; i = 1, 2, ..., n.$$

Let $N_{\mathcal{X}} = \mathbb{Z}^n$ be the lattice with standard basis $f_1, ..., f_n$, whereas $M_{\mathcal{X}} = N_{\mathcal{X}}^* = (\mathbb{Z}^n)^* = \mathbb{Z}^n$, with standard basis $e_1, ..., e_n$, which is the dual basis of $f_1, ..., f_n$, i.e., $e_i(f_j) = \delta_{ij}$.

Let $(\mathbb{P}^1)^n$ be the projective variety with coordinates $z_1, ..., z_n$. Define a subvariety $A_i \subset (\mathbb{P}^1)^n$ for each i by:

$$A_i = \begin{cases} \left(\bigcap_{b_{ij}>0} (z_j = \infty)\right) \bigcap (z_i = 0) \\ \left(\bigcap_{b_{ij}<0} (z_j = \infty)\right) \bigcap (z_i = 0) \end{cases} = \left(\bigcap_{b_{ij}\neq 0} (z_j = \infty)\right) \bigcap (z_i = 0)$$

Let X be the blowup of A_i, $i = 1, ..., n$, in $(\mathbb{P}^1)^n$ in some order. Since the exceptional divisor over $(\bigcap_{b_{ij}\neq 0}(z_j = \infty))\bigcap(z_i = 0)$ corresponds to the ray $\rho_{n+i} = \mathbb{R}_{\geq 0} \cdot (f_i - \sum_{k=1,k\neq i}^{n} |b_{ik}| \cdot f_k)$, so X is a toric variety with rays $\{\rho_1, \rho_2, , ..., \rho_{3n}\}$.

Remark 2.7 *X is only well-defined once we choose an order of the blowup of the A_i. Then X is a smooth toric variety. In general, for two different orders of blowups, the corresponding varieties X will agree in codimension one.*

We have a bijective divisor correspondence of divisors under a birational map which is an isomorphism up to codimension two. Suppose we have two varieties X_1 and X_2, and a birational map $f : X_1 \dashrightarrow X_2$, which is an isomorphism up to codimension two, i.e., we have $U \subset X_1$, $V \subset X_2$, $f : U \xrightarrow{\sim} V$ and cod $X_1 \setminus U \geq 2$, cod $X_2 \setminus V \geq 2$. Then we have a bijection between $\{$divisors on $X_1\}$ and $\{$divisors on $X_2\}$, by the following map:

$$D \longmapsto \overline{f(D \cap U)}.$$

After setting up these notations, we have the following theorem, which shows the $\mathcal{X}$ cluster variety for the root system Φ is isomorphic to an open subset of a toric variety up to codimension two.

A Dynkin diagram with no multiple edges is called simply laced, as are the corresponding Lie algebra and Lie group.

Theorem 2.8 *Let Φ be a simply laced root system, i.e., a type A, D, E root system. Let X be the toric variety corresponding to the root system Φ as defined above, and*

$U = X \setminus D$, where $D = \sum_{i=1}^n H_i + \sum_{i=1}^n B_{2n+i}$, $H_i = \overline{(\mathcal{X}^{e_i} = 1)}$ is the closure of a hypertorus, and B_{2n+i} is the toric boundary divisor corresponding to ray $\rho_{2n+i} = \mathbb{R}_{\geq 0} \cdot (-f_i)$ for $i = 1, 2, ..., n$. Then U is isomorphic to the $\mathcal{X}$ cluster variety for the root system Φ up to codimension two.

Proof. Let $\mathbf{s}$ be a seed in the cluster variety $\mathcal{X}$ such that every vertex is a "source" or a "sink". Following the notation from [GHKII], we consider the fan

$$\Sigma_{\mathbf{s},\mathcal{X}} = \{0\} \cup \{\mathbb{R}_{\geq 0} \cdot (-v_i), i = 1, ..., n\}.$$

Let $TV_{\mathbf{s},\mathcal{X}}$ be the toric variety defined by the fan $\Sigma_{\mathbf{s},\mathcal{X}}$. Let $\bar{D}_i$ be the toric boundary divisor corresponding to the ray $\mathbb{R}_{\geq 0} \cdot (-v_i)$ in the fan $\Sigma_{\mathbf{s},\mathcal{X}}$.

Let $w_i = X_i = \mathcal{X}^{e_i}$ be the $\mathcal{X}$-cluster variables of the cluster variety. Recall that $v_i = \{e_i, \cdot\} = \sum_{j=1}^n b_{ij} f_j$, so $\mathbb{R}_{\geq 0} \cdot (-v_i) = \mathbb{R}_{\geq 0} \cdot \sum_{b_{ij} \neq 0} (-f_j)$ corresponds to $\Gamma_i = \bigcap_{\{j | b_{ij} > 0\}} (w_j = \infty)$ or $\Gamma_i = \bigcap_{\{j | b_{ij} < 0\}} (w_j = 0)$, since for every i, we have $b_{ij} \geq 0, \forall j$, or $b_{ij} \leq 0, \forall j$.

Define $A_i \subset \Gamma_i$ by

$$A_i := \Gamma_i \cap (1 + \mathcal{X}^{e_i} = 0) = \Gamma_i \cap (w_i = -1),$$

and let $\pi : V \longrightarrow (\mathbb{P}^1)^n{}_{w_1,...,w_n}$ be the blow-up along $\bigcup_{i=1}^n \Gamma_i$.

Then $\mathbb{R}_{\geq 0} \cdot (-v_i)$ corresponds to the exceptional divisor for the blow up of this locus Γ_i, or to this locus itself if there exists a unique j such that $b_{ij} \neq 0$ (in this case, Γ_i is already a divisor). Note, in the simply laced case we are considering, $|b_{ij}| = 0$ or 1, so this is an ordinary blowup, not a weighted blowup.

Let F_i be the exceptional divisor for the blow-up of this locus Γ_i, so

$$\rho_{F_i} = \mathbb{R}_{\geq 0} \cdot (-v_i) = \mathbb{R}_{\geq 0} \cdot \sum_{b_{ij} \neq 0} (-f_j).$$

We define $W_i \subset F_i \subset V$ by

$$W_i := (1 + \mathcal{X}^{e_i} = 0) \cap F_i = (w_i = -1) \cap F_i = \begin{cases} \bigcap\limits_{b_{ij}>0} (w_j = 0) \cap (w_i = -1) \\ \bigcap\limits_{b_{ij}<0} (w_j = \infty) \cap (w_i = -1) \end{cases}$$

And let $\alpha : \tilde{V} \longrightarrow V$ be the blow-up along $\bigcup_{i=1}^{n} W_i$.

Let G_i be the exceptional divisor for blow-up of this locus W_i, so

$$\rho_{G_i} = \mathbb{R}_{\geq 0} \cdot (f_i - \sum_{b_{ij} \neq 0} f_j).$$

We have the following maps:

$$(\mathbb{P}^1)^n{}_{w_1,\ldots,w_n} \overset{\pi}{\longleftarrow} V \overset{\alpha}{\longleftarrow} \tilde{V}$$

We define

$$\theta^* z_j = z_j \circ \theta = \begin{cases} 1/w_j + 1 & \text{if } b_{ij} > 0 (b_{ji} < 0) \quad \text{for some} \quad i \\ w_j + 1 & \text{if } b_{ij} < 0 (b_{ji} > 0) \quad \text{for some} \quad i, \end{cases}$$

this gives an isomorphism

$$\theta : (\mathbb{P}^1)^n{}_{w_1,\ldots,w_n} \overset{\theta}{\longrightarrow} (\mathbb{P}^1)^n{}_{z_1,\ldots,z_n}.$$

Under this coordinate change, we have

$$\begin{cases} (w_j = 0) \longmapsto (z_j = \infty) \quad \text{and} \quad (w_j = \infty) \longmapsto (z_j = 1) & \text{if } b_{ij} > 0 \\ (w_j = 0) \longmapsto (z_j = 1) \quad \text{and} \quad (w_j = \infty) \longmapsto (z_j = \infty) & \text{if } b_{ij} < 0 \\ (w_j = -1) \longmapsto (z_j = 0) \quad \forall i \end{cases}$$

so

$$(\mathbb{P}^1)^n{}_{w_1,\ldots,w_n} \supset \Gamma_i = \begin{cases} \bigcap\limits_{b_{ij}>0} (w_j = 0) \\ \bigcap\limits_{b_{ij}<0} (w_j = \infty) \end{cases} \overset{\theta}{\longmapsto} \bigcap_{b_{ij} \neq 0} (z_j = \infty) \quad \text{for any} \quad i$$

$$V \supset F_i = \begin{cases} \bigcap\limits_{b_{ij}>0} (w_j = 0) \\ \\ \bigcap\limits_{b_{ij}<0} (w_j = \infty) \end{cases} \quad \stackrel{\theta}{\longmapsto} \underset{b_{ij}\neq 0}{\cap} (z_j = \infty) \quad \text{for any} \quad i$$

and

$$W_i = \begin{cases} \bigcap\limits_{b_{ij}>0} (w_j = 0) \cap (w_i = -1) \\ \\ \bigcap\limits_{b_{ij}<0} (w_j = \infty) \cap (w_i = -1) \end{cases} \quad \stackrel{\theta}{\longmapsto} \begin{cases} (\underset{b_{ij}>0}{\cap} (z_j = \infty)) \bigcap (z_i = 0) \\ \\ (\underset{b_{ij}<0}{\cap} (z_j = \infty)) \bigcap (z_i = 0) \end{cases}$$

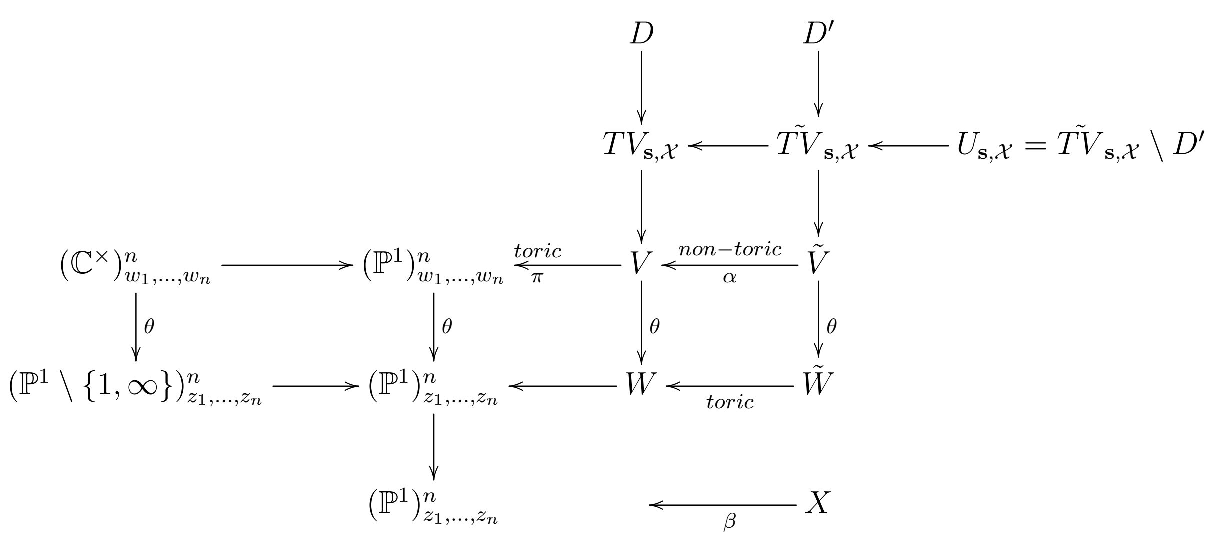
$$= (\underset{b_{ij}\neq 0}{\cap} (z_j = \infty)) \bigcap (z_i = 0)$$

Let W and $\tilde{W}$ be the corresponding varieties under the map θ, where W is the blow-up of $(\mathbb{P}^1)^n$ along $\bigcup_{i=1}^n \Gamma_i$, and $\tilde{W}$ is the blow-up of W along $\bigcup_{i=1}^n W_i$. We have the map

$$(\mathbb{P}^1)^n{}_{z_1,\dots,z_n} \stackrel{\pi}{\longleftarrow} W \stackrel{\alpha}{\longleftarrow} \tilde{W}.$$

Consider $A_i = \Gamma_i \cap (z_i = 0)$, and let $\beta : X \longrightarrow (\mathbb{P}^1)^n{}_{z_1,\dots,z_n}$ be the blow-up of $(\mathbb{P}^1)^n$ along $\bigcup_{i=1}^n A_i$.

So we have the following diagram,



Because V and $TV_{\mathbf{s},\mathcal{X}}$ have same rays $\mathbb{R}_{\geq 0} \cdot (-v_i) = \mathbb{R}_{\geq 0} \cdot \sum_{b_{ij}\neq 0}(-f_j)$, but V

has additional rays $\mathbb{R}_{\geq 0} \cdot f_j$ and $\mathbb{R}_{\geq 0} \cdot (-f_j)$, we can see, up to codimension two,

$$TV_{\mathbf{s},\mathcal{X}} = V \setminus (\cup_i (w_i = 0)' \bigcup \cup_i (w_i = \infty)').$$

Then we have

$$U_{\mathbf{s},\mathcal{X}} = \tilde{TV}_{\mathbf{s},\mathcal{X}} \setminus D' = \tilde{V} \setminus (\cup_i (w_i = 0)'' \bigcup \cup_i (w_i = \infty)'' \bigcup \cup_i F_i').$$

Under the map θ, up to codimension two,

$$U_{\mathbf{s},\mathcal{X}} = \tilde{W} \setminus (\cup_i (z_i = 1)'' \bigcup \cup_i (z_i = \infty)'' \bigcup \cup_i F_i').$$

On the other hand, since X and $\tilde{W}$ have same rays

$$\mathbb{R}_{\geq 0} \cdot f_i,\ \mathbb{R}_{\geq 0} \cdot (f_i - \sum_{k=1, k\neq i}^{n} |b_{ik}| \cdot f_k),\ \text{and}\quad \mathbb{R}_{\geq 0} \cdot (-f_i),\ i = 1, 2, ..., n,$$

but $\tilde{W}$ has additional rays $\mathbb{R}_{\geq 0} \cdot \sum_{b_{ij} \neq 0} (-f_j)$, we know up to codimension two

$$X = \tilde{W} \setminus \cup_i F_i'$$

Thus up to codimension two,

$$U_{\mathbf{s},\mathcal{X}} = X \setminus D = U,$$

where $D = \cup_i (z_i = 1)'' \bigcup \cup_i (z_i = \infty)''$, $(z_i = 1)$ corresponds to the hypertori $H_i = \overline{(\mathcal{X}^{e_i} = 1)}$, and $(z_i = \infty)$ corresponds to B_{2n+i}, which is the toric divisor associated to the ray generated by $\mathbb{R}_{\geq 0} \cdot (-f_i)$. We have $D = \sum_{i=1}^n H_i + \sum_{i=1}^n B_{2n+i}$.

We have $U_{\mathbf{s},\mathcal{X}} = \mathcal{X}$ up to codimension two by Lemma 2.6 .

So we have proved that U is isomorphic to the $\mathcal{X}$ cluster variety for the root system Φ up to codimension two.

The toric boundary divisors of X are $(z_i = 0)$, $(z_i = \infty)$ and the exceptional divisors of π. The divisor $(z_i = 0)$ corresponds to rays generated by $\mathbb{R}_{\geq 0} \cdot f_i$, the divisor ($z_i = \infty$) corresponds to rays generated by $\mathbb{R}_{\geq 0} \cdot (-f_i)$, and the exceptional divisors of π (the composition of blowups of $(\bigcap_{b_{ij} \neq 0} (z_j = \infty)) \bigcap (z_i = 0)$), corresponds to the rays generated by $f_i - \sum_{b_{ij} \neq 0} f_j$, for $i = 1, 2, ..., n$.

$\diamond$

Remark 2.9 *Why can we consider cluster varieties up to codimension two?*

Since the ring of functions on a variety is determined away from a set of codimension two, we can study the $\mathcal{X}$- and $\mathcal{A}$-cluster algebras by studying the corresponding cluster varieties up to codimension two.

Suppose X and Y are normal varieties and the map $X \dashrightarrow Y$ is an isomorphism up to codimension two. Then we have $Cl\ X \simeq Cl\ Y$. Suppose $D_X \in Cl\ X$, we have the corresponding $D_Y \in Cl\ Y$, then $H^0(\mathcal{O}_X(D_X)) \simeq H^0(\mathcal{O}_Y(D_Y))$ by the Hartogs type property, since the meromorphic functions on a prescribed set are same.

In particular, the Cox rings are isomorphic.

Lemma 2.10 *Let X be the toric variety in Theorem 2.8. The fan of X has rays $\rho_i = \mathbb{R}_{\geq 0} \cdot v_i$ for $i = 1, ..., 3n$. Let B_i be the toric boundary divisor corresponding to the ray ρ_i.*

Let $U = X \setminus D$, where $D = \sum_{i=1}^n H_i + \sum_{i=1}^n B_{2n+i}$, $H_i = \overline{(\mathcal{X}^{e_i} = 1)}$ is the hypertori.

Then we have

$$\overline{(\mathcal{X}^{e_i} = 0)} = B_i + B_{n+i}$$

and

$$H_i = \overline{(\mathcal{X}^{e_i} = 1)} \sim \overline{(\mathcal{X}^{e_i} = 0)} \sim \overline{(\mathcal{X}^{e_i} = \infty)},$$

In particular,

$$K_X + D \sim 0,$$

and we have that (X, D) is a log Calabi-Yau pair.

Proof. We have $H_i = \overline{(\mathcal{X}^{e_i} = 1)}$ by definition. Also, from the character map $X \xrightarrow{\mathcal{X}^{e_i}} \mathbb{P}^1$, we know

$$\overline{(\mathcal{X}^{e_i} = 1)} \sim \overline{(\mathcal{X}^{e_i} = 0)} \sim \overline{(\mathcal{X}^{e_i} = \infty)}.$$

Thus

$$H_i = \overline{(\mathcal{X}^{e_i} = 1)} \sim \overline{(\mathcal{X}^{e_i} = 0)} \sim \overline{(\mathcal{X}^{e_i} = \infty)}.$$

Now let's compute $(\mathcal{X}^{e_i} = 0)$ and $(\mathcal{X}^{e_i} = \infty)$ explicitly.

In general, if X is a toric variety and D is an irreducible component of the toric boundary, then D corresponds to a ray $\rho = \mathbb{R}_{\geq 0} \cdot v$. Let $\mathcal{X}^m \in \mathrm{Hom}(T, \mathbb{C}^\times)$ be a character on X, $m \in M$. Then $\mathrm{ord}_D(\mathcal{X}^m) = \langle m, v \rangle \in \mathbb{Z}$.

In particular,

$$(\mathcal{X}^m) = (\mathcal{X}^m = 0) - (\mathcal{X}^m = \infty) = \sum_{i=1}^{3n} \langle m, v_i \rangle \cdot B_i,$$

where B_i is the divisor corresponds to $\rho_i = \mathbb{R}_{\geq 0} \cdot v_i$.

Now in our case, define

$$n_+ = \begin{cases} n, & \text{if} \quad n \geq 0 \\ 0, & \text{if} \quad n < 0 \end{cases}$$

and

$$n_- = \begin{cases} n, & \text{if} \quad n \leq 0 \\ 0, & \text{if} \quad n > 0 \end{cases}$$

Then we have

$$(\mathcal{X}^{e_i} = 0) = \sum_{j=1}^{3n} \langle e_i, v_j \rangle_+ B_j = B_i + B_{n+i},$$

and

$$(\mathcal{X}^{e_i} = \infty) = \sum_{j=1}^{3n} |(\langle e_i, v_j \rangle_-)| B_j = B_{2n+i} + \sum_{i,j \text{ an edge}} B_{n+j}.$$

This is because $v_j = f_j$, $v_{n+j} = f_j - \sum_{k=1, k \neq j}^{n} |b_{jk}| \cdot f_k$, $v_{2n+j} = -f_j$, $i = 1, ..., n$.

From that, we have

$$\sum_{i=1}^{n} H_i \sim \sum_{i=1}^{n} (B_i + B_{n+i}).$$

So we have

$$D = \sum_{i=1}^{n} H_i + \sum_{i=1}^{n} B_{2n+i} \sim \sum_{i=1}^{n} (B_i + B_{n+i}) + \sum_{i=1}^{n} B_{2n+i} = B.$$

Since $B = \sum_{i=1}^{3n} B_i$ is the toric boundary of X, then $K_X + B \sim 0$.

Thus

$$K_X + D \sim K_X + B \sim 0$$

.

Since X is smooth and D is a normal crossing, we see that (X, D) is a log Calabi-Yau pair. ◇

We can generalize Theorem 2.8 to any finite type root system. It is not only true for type A, D, E root systems, but also true for type B, C, F, G root systems. But for the simply laced case (A, D, E root system), $v_i = \{e_i, \cdot\} = \sum b_{ij} f_j$, $b_{ij} = 0, \pm 1$, thus $\text{ind}(v_i) = 1$; for other cases, $\text{ind}(d_i v_i) = d > 1$. (See [FZ03] and [GHKII] for definitions of d_i.) If we blow up Z_i with multiplicity $d > 1$, then we will get a singular locus in X with transverse slice an A_{d-1} singularity.

Theorem 2.11 *Let Φ be any root system. Let X be the toric variety corresponding to the root system Φ with rays $\{\rho_1, \rho_2, , ..., \rho_{3n}\}$, $\rho_i = \mathbb{R}_{\geq 0} \cdot v_i$ for $i = 1, .., 3n$, where*

$$v_i = f_i, \ v_{n+i} = f_i - \sum_{k=1, k \neq i}^{n} |b_{ik}| \cdot f_k, \ v_{2n+i} = -f_i, \ i = 1, 2, ..., n.$$

Let $U = X \setminus D$, where $D = \sum_{i=1}^n H_i + \sum_{i=1}^n B_{2n+i}$, $H_i = \overline{(\mathcal{X}^{e_i} = 1)}$ is the closure of a hypertorus, and B_{2n+i} is the toric boundary divisor corresponding to ray $\rho_{2n+i} = \mathbb{R}_{\geq 0} \cdot (-f_i)$ for $i = 1, 2, ..., n$. Then U is isomorphic to the $\mathcal{X}$ cluster variety for the root system Φ up to codimension two.

The proof is very similar to the proof of Theorem 2.8. We will omit the proof.

Remark 2.12 *Any bipartite graph will work. We don't need Φ to be a root system of finite type. A bipartite graph is a graph whose vertices can be divided into two disjoint sets U and V such that every edge connects a vertex in U to one in V. Equivalently, a bipartite graph is a graph that does not contain any odd-length cycles. It is equivalent to $I \cup J = \{1, 2, ..., n\}$, $a_{ij} = 0$ when $i, j \in I$, or $i, j \in J$.*

For any bipartite graph, we can define the $\mathcal{X}$-cluster variety associated to the generalized Cartan matrix $A = (a_{ij})$ and the analogue of Theorem 2.8 holds.

2.3 Toric degeneration

We treat the cluster variety as a toric variety with a non-toric boundary divisor removed. We can also degenerate a cluster variety to a torus. The following proposition gives a toric degeneration from a $\mathcal{X}$-cluster variety U to a torus.

Theorem 2.13 *Let U be an $\mathcal{X}$-cluster variety of finite type for the root system Φ. X is the projective toric variety constructed in Theorem 2.8. It is a blowup of $(\mathbb{P}^1)^n$. Let $D \subset X$ be a non-toric divisor of X defined in Theorem 2.8, where $D = \sum_{i=1}^n H_i + \sum_{i=1}^n B_{2n+i}$.*

Then there exist families $(X \times \mathbb{A}^n, \mathcal{D}_t)$ such that $\mathcal{D}_{0,...,0}$ is the toric boundary of X and $\mathcal{D}_{1,...,1}$ is the divisor D. Let $(\mathbb{C}^\times)^n$ be the torus inside $\mathbb{A}^n$, then the fibers $(\mathcal{X}_t, \mathcal{D}_t)$ over points $t \in (\mathbb{C}^\times)^n$ are isomorphic.

Proof.

For $(t_1, t_2, ..., t_n) = t \in \mathbb{A}^n$, we define the family $\mathcal{D}_t = \sum_{i=1}^n \mathcal{H}_{i,t} + \sum_{i=1}^n \mathcal{B}_{2n+i}$, where $\mathcal{H}_{i,t} = \overline{(\mathcal{X}^{e_i} = t_i)}$ is the hypertorus, and $\mathcal{B}_{2n+i} = B_{2n+i} \times \mathbb{A}^n$ for $i = 1, ..., n$.

Now $\mathcal{D}_t = \sum_{i=1}^n \overline{(\mathcal{X}^{e_i} = t_i)} + \sum_{i=1}^n B_{2n+i} \times \mathbb{A}_n$ is a subset of $X \times \mathbb{A}^n_{t_1,...,t_n}$.

We need to check that $\mathcal{D}_{0,...,0}$ is the toric boundary B and $\mathcal{D}_{1,...,1}$ is the divisor D:

By Lemma 2.10, we know $\overline{(\mathcal{X}^{e_i} = 0)} = B_i + B_{n+i}$, thus

$$\begin{aligned}\mathcal{D}_{0,...,0} &= \sum_{i=1}^n \mathcal{H}_{i,0} + \sum_{i=1}^n \mathcal{B}_{2n+i,0}\\ &= \sum_{i=1}^n \overline{(\mathcal{X}^{e_i} = 0)} + \sum_{i=1}^n \mathcal{B}_{2n+i,0}\\ &= \sum_{i=1}^n (B_i + B_{n+i}) + \sum_{i=1}^n B_{2n+i}\\ &= \sum_{i=1}^{3n} B_i = B,\end{aligned}$$

and

$$\begin{aligned}\mathcal{D}_{1,...,1} &= \sum_{i=1}^n \mathcal{H}_{i,1} + \sum_{i=1}^n \mathcal{B}_{2n+i,1}\\ &= \sum_{i=1}^n \overline{(\mathcal{X}^{e_i} = 1)} + \sum_{i=1}^n \mathcal{B}_{2n+i,1}\\ &= \sum_{i=1}^n H_i + \sum_{i=1}^n B_{2n+i}\\ &= D.\end{aligned}$$

Next, we show $(X \times \mathbb{A}^n, \mathcal{D}_t)$ is actually a flat family.

Since X is a toric variety, X is Cohen-Macaulay. So $X \times \mathbb{A}^n$ is also Cohen-Macaulay. We have the following map:

$$\begin{array}{ccc} X \times \mathbb{A}^n & \supset & \mathcal{D}_t = \sum_{i=1}^n \mathcal{H}_{i,t} + \sum_{i=1}^n \mathcal{B}_{2n+i} \\ \downarrow{\scriptstyle pr_2} & \swarrow & \\ \mathbb{A}^n & & \end{array}$$

We need to prove each component of $\mathcal{D}_t$ is flat over $\mathbb{A}^n$, that is to prove both $\mathcal{B}_{2n+i}$ and $\mathcal{H}_{i,t}$ are flat over $\mathbb{A}^n$.

We have $\mathcal{B}_{2n+i} = B_{2n+i} \times \mathbb{A}^n \to \mathbb{A}^n$ is flat since this is a trivial bundle.

For the map $f : \mathcal{H}_{i,t} = \overline{(\mathcal{X}^{e_i} = t_i)} \longrightarrow \mathbb{A}^n$, since X is smooth and $\mathcal{H}_{i,t}$ has codimension 1, we know $\mathcal{H}_{i,t}$ is locally defined by one equation. So $\mathcal{H}_{i,t}$ is also Cohen-Macaulay.

We also have that $\mathbb{A}^n$ is smooth, and for the map $f : \mathcal{H}_{i,t} \to \mathbb{A}^n$: the fibers of f have the same dimension $n-1$. Thus $f : \mathcal{H}_{i,t} \to \mathbb{A}^n$ is flat.

Here we use the following theorem.

(X, Y are two varieties, X is Cohen-Macaulay, Y is smooth, we have a map $f : X \to Y$. Then if the fibers of f have the same dimension, f is flat)

(Ref: Hartshorne ExIII.10.9, p276)

So each component of $\mathcal{D}_t$ is flat over $\mathbb{A}^n$.

Thus $(X \times \mathbb{A}^n, \mathcal{D}_t)$ is a flat family.

Let $\mathcal{U}_t = X \times \mathbb{A}^n \backslash \mathcal{D}_t$. So we have the map

$$\mathcal{U}_t = X \times \mathbb{A}^n \backslash \mathcal{D}_t \xrightarrow{f} \mathbb{A}^n.$$

Then $\mathcal{U}_{0,\ldots,0} = f^{-1}(0) = X \backslash B = T \simeq (\mathbb{C}^\times)^n$, and $\mathcal{U}_{1,\ldots,1} = X \backslash D = U$.

For any $t \in (\mathbb{C}^\times)^n$, we have $\mathcal{U}_t = f^{-1}(t) \simeq U$.

Thus the fiber $(\mathcal{X}_t, \mathcal{D}_t)$ over any point in $(\mathbb{C}^\times)^n$ are isomorphic, and we have described a toric degeneration from a $\mathcal{X}$-cluster variety U to a torus.

$\diamond$

Remark 2.14 *Compare to [GHKK], in section 5. They give a degeneration from an $\mathcal{A}$ cluster variety to a torus $(\mathbb{C}^\times)^n$, and use it to prove the Fock-Goncharov dual basis conjecture ([FG06], Conjecture 4.3). We can get a similar degeneration from* *an $\mathcal{X}$ cluster variety to a torus $(\mathbb{C}^\times)^n$ from the degeneration of an $\mathcal{A}$ cluster variety in [GHKK], which is a generalization of our case.*

2.4 $-K_X$ is not nef in general

We know from Theorem 2.8 that X is a toric variety with rays $\{\rho_1, \rho_2, , ..., \rho_{3n}\}$, where $\rho_i = \mathbb{R}_{\geq 0} \cdot v_i$, and $v_i = f_i$, $v_{n+i} = f_i - \sum_{k=1,k\neq i}^{n} |b_{ik}| \cdot f_k$, $v_{2n+i} = -f_i$, $i = 1, 2, ..., n$. Now we can consider a polytope P which is a convex hull of these vertices $\{v_1, v_2, ..., v_{3n}\}$. Let's first review the definition of a polytope.

Definition 2.15 *A* polytope *in $N_\mathbb{R}$ is a set of the form*

$$P = Conv(S) = \{\sum_{v\in S} \lambda_v v \mid \sum_{v \in S} \lambda_v = 1, 0 \leq \lambda_v \leq 1\} \subset N_\mathbb{R}$$

where $S \subset N_\mathbb{R}$ is finite. We say that P is the Convex hull *of S.*

Let the polytope $P := Conv(v_1, ...v_{3n})$ be the Convex hull of $\{v_1, ...v_{3n}\}$. We can see each v_i is a vertex of the polytope:

Lemma 2.16 *Let $\rho_i = \mathbb{R}_{\geq 0} \cdot v_i$, where $v_i \in N$ is a primitive generator of N, $v_i = f_i$, $v_{n+i} = f_i - \sum_{k\neq i} |b_{ik}| f_k$, $v_{2n+i} = -f_i$, $i = 1, ..., n$, and $f_1, f_2, ..., f_n$ is the standard basis of $\mathbb{Z}^n$. Then $P := Conv(v_1, ...v_{3n}) \subset N_\mathbb{R}$ is a convex polytope containing 0 in its interior and with vertices $\{v_1, ...v_{3n}\}$.*

Proof.

See appendix.

◇

Now we can consider the toric variety corresponding to the polytope P. Let Σ_Z be the face fan of the convex polytope P in N, which consists of the cones over the faces of P. Let Z be the toric variety with fan Σ_Z. We call Z the toric variety corresponding to a polytope P. This defines a compact toric variety Z such that X is isomorphic to Z up to codimension two, and Z is probably singular.

Let $-K_Z$ be the anti-canonical divisor of Z, then $-K_Z$ is an ample divisor and Z is a toric Fano variety:

Lemma 2.17 *([Reid], Proposition 4.3) Let P be a convex polytope in $\mathrm{N}_{\mathbb{R}}$ containing 0 in its interior. Suppose every vertex of P is a primitive generator of the lattice N. Let Σ_Z be the* face fan *of the convex polytope P in N, which consists of the cones over the faces of P.*

Let Z be the toric variety with fan Σ_Z, then Z is a possibly singular Fano variety and the anti-canonical divisor $-K_Z$ is $\mathbb{Q}$-Cartier and ample.

Proof. First, let's prove $-K_Z$ is a $\mathbb{Q}$-Cartier divisor.

Let $\sigma \in \Sigma_Z$ be a maximal cone in the fan. So $\dim \sigma = \dim_{\mathbb{C}} Z = n$, $\sigma = \langle v_{i_1}, ..., v_{i_r} \rangle$ and $v_{i_1}, ..., v_{i_r}$ lie in same hyperplane. So there exists $m \in M = N^*$ such that $\langle m, v_{i_j} \rangle = c \in \mathbb{N}$ for $j = 1, ..., r$.

We have an open subset $U = Spec\mathbb{C}[\sigma^* \cap M] \subset Z$ corresponding to σ. Then

$$(\mathcal{X}^m)|_U = c \cdot (D_{i_1} + \cdots + D_{i_r})|_U = c \cdot (-K_Z)|_U.$$

Because Z is covered by U, we know $-K_Z$ is a $\mathbb{Q}$-Cartier divisor.

Second, we prove $-K_Z$ is an ample divisor. It is enough to show for any toric 1-strata $C \subset Z$, we have $(-K_Z) \cdot C > 0$.

For any toric 1-strata $C \subset Z$, it corresponds to a codimension one face τ, which is the intersection of two maximal cones, i.e., $\tau = \sigma_1 \cap \sigma_2$. Then we can check that $(-K_Z) \cdot C > 0$ by convexity of the polytope P, see [[Reid], Proposition 4.3]. ◇

We have a natural birational map π from X to Z, which is an isomorphism up to codimension two, since the toric varieties X and Z have the same rays. A natural question is: when is π a morphism?

If π is a morphism, then $\pi : X \longrightarrow Z$ is the resolution of singularities of Z, and the exceptional locus is the union of curves $C \subset X$ such that $-K_X \cdot C = 0$. We can see π is a morphism if and only if the anti-canonical divisor $-K_X$ is a nef divisor:

Lemma 2.18 *Let X be the toric variety defined in Theorem 2.8, which is a blowup of $(\mathbb{P}^1)^n$. Let Z be the toric variety corresponding to the polytope P, and π is the natural birational map π from X to Z, which is an isomorphism up to codimension two.*

$$\begin{array}{ccc} X & \overset{\pi}{\dashrightarrow} & Z \\ \downarrow & & \\ (\mathbb{P}^1)^n & & \end{array}$$

Then $\pi : X \dashrightarrow Z$ is a morphism if and only if $-K_X$ is nef.

Proof. Suppose $\pi : X \longrightarrow Z$ is a morphism, then $-K_X = \pi^*(-K_Z)$. Let C be any curve in X. By the projection formula, we have $C \cdot \pi^*(-K_Z) = \pi_* C \cdot (-K_Z)$. Since $-K_Z$ is nef, we have

$$C \cdot (-K_X) = C \cdot \pi^*(-K_Z) = \pi_* C \cdot (-K_Z) \geq 0.$$

Thus $-K_X$ is a nef divisor.

Conversely, suppose $-K_X$ is a nef divisor. For any $\sigma \in \Sigma_X$, let $\sigma = \langle v_1, ..., v_r \rangle_{\mathbb{R}\geq 0}$, where $v_1, ..., v_r$ are primitive integral generators of rays of Σ. Let $\sigma' = Conv(0, v_1, ..., v_r)$. By a similar argument to Lemma 2.18, we know $\bigcup \sigma'$ is convex. Now $\bigcup \sigma' = P := Conv(v_1, v_2, ..., v_{3n})$, v_i are primitive generators of all rays of Σ. Thus Σ_X refines the face fan Σ_Z of P which is the fan of Z. So, $\pi : X \dashrightarrow Z$ is a morphism.

◇

Whether $\pi : X \dashrightarrow Z$ is a morphism or not can be checked torically, i.e., $-K_X$ is nef if for any toric 1-strata $C \subset X$, we have $-K_X \cdot C \geq 0$.

If $-K_X$ is nef, then π is a morphism and the exceptional locus is the union of toric 1-strata $C \subset X$ such that $-K_X \cdot C = 0$.

See the following two lemmas for detailed computations.

Lemma 2.19 *(Kleiman Cone Theorem) ([KM], Theorem 1.18) Let X be a smooth projective variety over $\mathbb{C}$, and A a divisor on X. Let*

$$\overline{NE}(X) = \overline{Curv}(X) = \overline{\{\Sigma a_i[C_i] | C_i \subset X \ curves, a_i \in \mathbb{R}_{\geq 0}\}} \subset H_2(X, \mathbb{R}).$$

Then A is ample if and only if $A \cdot x > 0$ for any $x \in \overline{Curv}(X) \setminus \{0\}$.

In particular, if $\overline{Curv}(X) = \langle [C_1], ..., [C_r] \rangle_{\mathbb{R}_{\geq 0}}$ for some curves $C_1, ..., C_r \subset X$, then A is ample if and only if $A \cdot C_i > 0$ for all i

Lemma 2.20 *Let X be a n dimensional smooth toric variety. The cone of curves of a toric variety is generated by the toric 1-strata.([CLS11], Theorem 6.3.20b.) Let $C \subset X$ be a 1-strata of the toric boundary.*

Then $-K_X \cdot C \geq 0$ if and only if v_{n+1} lies below the affine hyperplane through $\{v_1, ..., v_{n-1}, v_n\}$, where $\langle v_1, ..., v_{n-1}, v_n \rangle_{\mathbb{R}_{\geq 0}}$ and $\langle v_1, ..., v_{n-1}, v_{n+1} \rangle_{\mathbb{R}_{\geq 0}}$ are two maximal cones of the toric variety, and C corresponding to the intersection of these two cones, i.e., $\langle v_1, ..., v_{n-1} \rangle_{\mathbb{R}_{\geq 0}}$.

Equivalently, if $v_n + v_{n+1} = \sum_{i=1}^{n-1} b_i v_i$, then $-K_X \cdot C \geq 0$ if only if $\sum_{i=1}^{n-1} b_i \leq 2$.

Proof.

Suppose $C \subset X$ is a curve in X, i.e., $C \simeq \mathbb{P}^1$, then $\mathcal{N}_{C/X} = \bigoplus_{i=1}^{n-1} \mathcal{O}(-b_i)$, and $-K_X \cdot C = 2 - \sum_{i=1}^{n-1} b_i \geq 0$.

By the adjunction formula, we have

$$K_X \cdot C + \deg \det \mathcal{N}_{C/X} = \deg(K_C) = 2g - 2 = -2.$$

We also have $\deg(-K_C) = 2 - 2g = 2$.

Thus

$$-K_X \cdot C = 2 + \deg \det \mathcal{N}_{C/X} = 2 - \sum_{i=1}^{n-1} b_i.$$

so $-K_X \cdot C \geq 0$ if only if $\sum_{i=1}^{n-1} b_i \leq 2$.

Let $m = v_1^* + ... + v_n^*$, then $(m = 1)$ is the affine hyperplane through $v_1, ..., v_{n-1}, v_n$. We have

$$\langle m, v_{n+1} \rangle = \langle m, \sum_{i=1}^{n-1} b_i v_i - v_1 \rangle = \sum_{i=1}^{n-1} b_i - 1.$$

So v_{n+1} lies below the affine hyperplane through the vertices $v_1, ..., v_{n-1}, v_n$ if and only if $\langle m, v_{n+1} \rangle \leq 1$, i.e., $\sum_{i=1}^{n-1} b_i \leq 2$.

$\diamond$

As in Theorem 2.8, we know that X is the blowup of n subvarieties in $(\mathbb{P}^1)^n$ in some order. To analyze the variety X explicitly, we review some basics about blow-up of a subvariety.

Blowing up a subvariety of a variety can be very complicated, but blowing up a subvariety of a toric variety is very easy to understand. It corresponds to a star subdivision of the fan.

Suppose X is a smooth toric variety with fan Σ, and Z be a smooth subvariety of X, which is a toric strata corresponds to a cone $\sigma \in \Sigma$. Let $\tilde{X}$ be the blowup of X along the subvariety Z:

$$\tilde{X} = Bl_Z X \xrightarrow{\pi} X.$$

Let $\tilde{\Sigma}$ be the fan of $\tilde{X}$.

If Z is a single point, then the corresponding cone σ is a maximal cone. Suppose $\sigma = \langle v_1, ..., v_n \rangle_{\mathbb{R}_{\geq 0}}$, where $v_1, ..., v_n$ are basis of $N \simeq \mathbb{Z}^n$, then the blowup of X along

Z is given by adding a ray

$$\rho = \mathbb{R}_{\geq 0} \cdot \sum_{i=1}^{n} v_i = \mathbb{R}_{\geq 0} \cdot (v_1 + ... + v_n),$$

and subdividing the cone σ.

For every facet F of σ, we get a new cone

$$\tau_F = \langle \rho, F \rangle.$$

So we get n new maximal cones

$$\tau_i = \langle v_1, ..., \hat{v}_i, ..., v_n, \sum_{i=1}^{n} v_i \rangle_{\mathbb{R}_{\geq 0}},\ \ i = 1, ..., n$$

If Z is an arbitrary toric strata, then it will be a bit more complicated.

Suppose Z has codimension r, then the corresponding cone $\sigma =< v_1, ..., v_r >_{\mathbb{R}_{\geq 0}}$. The blowup corresponds to adding a ray

$$\rho = \mathbb{R}_{\geq 0} \cdot \sum_{i=1}^{r} v_i = \mathbb{R}_{\geq 0} \cdot (v_1 + ... + v_r),$$

and subdividing σ as before.

For any maximal cone τ containing σ,

$$\sigma = \langle v_1, ..., v_r \rangle_{\mathbb{R}_{\geq 0}} \subset \tau = \langle v_1, ..., v_n \rangle_{\mathbb{R}_{\geq 0}},$$

where $v_1, ..., v_n$ are basis of $N \simeq \mathbb{Z}^n$. The blowup is given by subdividing τ by the cones

$$F_{\sigma'} =< v_{r+1}, ..., v_n, \sigma' >_{\mathbb{R}_{\geq 0}},$$

where σ' is a maximal cone in the subdivision of σ. We also have

$$F_{\sigma'} \cap F_{\sigma''} = F_{\sigma' \cap \sigma''}.$$

To be explicit, for any maximal cone τ containing σ, we get r new cones

$$F_{\tau,i} =< v_1, ..., \hat{v}_i, ...v_r, \sum_{i=1}^{r} v_i, v_{r+1}, ..., v_n >_{\mathbb{R}_{\geq 0}}, \quad i = 1, 2, ..., r.$$

By the previous discussion and Lemma 2.20, we can write a program in PARI/GP to check if $-K_X$ is not nef for our toric variety X defined in Theorem 2.8, which is a sequence of blowups of n loci in $(\mathbb{P}^1)^n$ corresponding to a root system Φ.

First, we can blow up n loci in $(\mathbb{P}^1)^n$ to get the toric variety X, and find all the maximal cones in the fan of X. For any two maximal cones

$$\langle v_1, ..., v_{n-1}, v_n \rangle_{\mathbb{R}_{\geq 0}} \quad \text{and} \quad \langle v_1, ..., v_{n-1}, v_{n+1} \rangle_{\mathbb{R}_{\geq 0}},$$

which share a face of codimension one, we can compute $\sum_{i=1}^{n-1} b_i$ as in Lemma 2.20. We list the numbers $\sum_{i=1}^{n-1} b_i$ for each pair of two maximal cones.

If all the numbers are ≤ 2, then $-K_X \cdot C \geq 0$ for any toric 1-strata C, so $-K_X$ is nef.

If for some maximal cones

$$\langle w_1, ..., w_{n-1}, w_n \rangle_{\mathbb{R}_{\geq 0}} \quad \text{and} \quad \langle w_1, ..., w_{n-1}, w_{n+1} \rangle_{\mathbb{R}_{\geq 0}},$$

we have $\sum_{i=1}^{n-1} b_i = a > 2$, then for the curve C corresponding to the intersection of the two maximal cones, i.e., $\langle w_1, ..., w_{n-1} \rangle_{\mathbb{R}_{\geq 0}}$, we have

$$-K_X \cdot C = 2 - a < 0,$$

so $-K_X$ is not nef.

See the appendix for the details of the program.

Thus $-K_X$ is not nef in general, i.e., $\pi : X \dashrightarrow Z$ is not a morphism in general cases, see the following examples.

Example 1 *Let X_{D_4} be blow up of 4 loci in $(\mathbb{P}^1)^4$ corresponding to the root system D_4, then $-K_{X_{D_4}}$ is not nef.*

Proof. Using PARI/GP, we can see there are four toric 1-strata C in X_{D_4} such that $-K_X \cdot C < 0$, and they are disjoint.

For example, we have two maximal cones

$$\sigma_1 = \langle -f_1, f_2, -f_3, f_4 \rangle_{\mathbb{R}_{\geq 0}} \quad \text{and} \quad \sigma_2 = \langle -f_1, f_2, -f_3, f_2 - f_1 - f_3 - f_4 \rangle_{\mathbb{R}_{\geq 0}},$$

then $\tau = \sigma_1 \cap \sigma_2 = \langle -f_1, f_2, -f_3 \rangle_{\mathbb{R}_{\geq 0}}$. Let C be the toric 1-strata corresponding to τ.

Then

$$f_4 + (f_2 - f_1 - f_3 - f_4) = 1 \cdot (-f_1) + 1 \cdot f_2 + 1 \cdot f_3,$$

thus $\mathcal{N}_{C/X} = \mathcal{O}(-1) \oplus \mathcal{O}(-1) \oplus \mathcal{O}(-1)$.

By the adjunction formula we have

$$K_X \cdot C + \deg\ \mathcal{N}_{C/X} = \deg\ K_C = 2g - 2 = -2,$$

so $-K_X \cdot C = -1$.

Thus $-K_X$ is NOT nef.

◇

Example 2 *We have the following calculations for A_3, A_4, D_5 root system, see the appendix for explicit coding and computation.*

1. *For the root system A_3, we can see there are no toric 1-strata C in X_{A_3} with $(-K) \cdot C < 0$, so $-K$ is nef for A_3. Then $\pi : X_{A_3} \dashrightarrow Z_{A_3}$ is a morphism.*

2. *For the root system A_4, similarly to D_4, we also have $-K$ is not nef and there is one toric 1-strata C such that $-K \cdot C < 0$.*

3. *For the root system D_5, we also have $-K$ is not nef and there are seven toric 1-strata C such that $(-K) \cdot C = -1$ and one toric 1-strata C such that $(-K) \cdot C = -2$.*

CHAPTER 3

FANO COMPACTIFICATION OF CLUSTER VARIETY

3.1 Toric minimal model program

In Chapter 2, we showed the cluster variety $\mathcal{X}$ is isomorphic to a toric variety with some boundary divisors and hypertori removed up to codimension two. In this Chapter, we will construct a nice "compactification" of $\mathcal{X}$.

Here is the explicit requirement of the compactification.

Given an cluster variety $U = \mathcal{X}$, we want a nice "compactification", i.e., a pair (X, D) such that:

1. $U = X \setminus D$ up to codimension two.

2. $K_X + D = 0$.

3. (X, D) has mild singularities.

We will give an introduction to the minimal model program (MMP), and then use a revised version of it, which is called the D-minimal model program, to get a nice compactification.

Suppose X is a smooth projective variety, then we have a sequence of birational transformations

$$X = X_0 \dashrightarrow X_1 \dashrightarrow X_2 \dashrightarrow \cdots \dashrightarrow X_n = Y.$$

Each step is a birational map, which is either a divisorial contraction or a flipping contraction.

A divisorial contraction is a morphism φ which contracts an irreducible divisor.

A flipping contraction contracts a locus of codimension ≥ 2, and then flips it. We have a diagram

$$\begin{array}{ccc} X & \overset{\varphi}{\dashrightarrow} & X^+ \\ \downarrow f & \swarrow g & \\ Z & & \end{array}$$

such that, $E \subset X$, $F \subset X^+$ are loci of codimension ≥ 2, and $X \setminus E \simeq X^+ \setminus F$. If f contracts a curve C to a point, and g contracts a curve Γ to a point, then we have $K_X \cdot C < 0$ and $K_{X^+} \cdot \Gamma > 0$.

Finally, the end result Y has either K_Y nef or $\exists\ h : Y \to S$, dim $S <$ dim Y, where if $C \subset Y$ is a curve such that $h(C)$ is a point, then $K_Y \cdot C < 0$.

But in our case, X is a projective toric variety, which is birationally equivalent to $(\mathbb{P}^1)^n$, so we cannot get a Y such that the canonical divisor K_Y is nef.

Instead, we will use the so-called D-minimal model program to make $D = -K_X$ nef, and get a sequence of birational maps. We modify the MMP for the toric case by replacing K_X with an anticanonical divisor $D = -K_X$. Then we can make $D = -K_X$ nef using elementary flips and divisorial contractions.

Then we can consider the sequence of birational modifications, which are isomorphisms in codimension one, starting with a blowup X of $(\mathbb{P}^1)^n$, through a toric variety Y such that $-K_Y$ is nef, and ending with a toric variety Z such that $-K_Z$ is ample, where Z is the toric variety defined in Lemma 2.18 corresponding to the polytope P.

$$\begin{array}{ccccc} X & \dashrightarrow & Y & \longrightarrow & Z \\ \downarrow & & & & \\ (\mathbb{P}^1)^n & & & & \end{array}$$

The sequence of birational modifications $X \dashrightarrow Y$ has a combinatorial description in terms of the secondary fan associated with the set of rays in the fan of X.

We can find the procedure in David Cox's book Toric Varieties [CLS11]. It is a special case of the Procedure 15.5.5 on p. 776–798 and the explicit algorithm in the proof of Proposition 15.5.6 on p. 777–799. We apply this algorithm with $D = -K_X$ to get the sequence of birational modifications from X to Y. Then $-K_Y$ is nef and defines a morphism $Y \longrightarrow Z$ which is birational and an isomorphism in codimension one, and is determined by a canonical coarsening of the fan of Y. Namely, the fan Σ_Z consists of the cones over the faces of the convex hull of the primitive generators of the rays in the fan Σ_Y of Y. In our case, $Y \longrightarrow Z$ is an isomorphism in codimension one because every ray of Σ_Y defines a vertex of P. This follows from the Lemma 2.16.

We state the Procedure 15.5.5 and Proposition 15.5.6 for the reader's convenience.

Proposition 3.1 *[CLS11] Let X_Σ be simplicial and projective, and let D be a Weil divisor on X_Σ. Then do the following steps:*

a) If D is nef, then stop.

b) If D is not nef, then by the toric cone theorem, there is an extremal ray R with $D \cdot R < 0$. Let $\phi : X_\Sigma \to X_{\Sigma_0}$ be the corresponding extremal contraction.

c) If ϕ is a fibering contraction, then stop.

d) If ϕ is a divisorial contraction, replace X_Σ and D with X_{Σ_0} and the birational transform $\phi_* D$. Note that X_{Σ_0} is simplicial and projective with support $|\Sigma_0| = |\Sigma|$, and $\phi_* D$ is the pushforward of D. Return to step (a) and continue.

e) If ϕ is a flipping contraction, then we have the flip

$$\begin{array}{ccc} X_\Sigma & \overset{\psi}{\dashrightarrow} & X_{\Sigma'} \\ \downarrow{\scriptstyle \phi'} & \swarrow{\scriptstyle \phi'} & \\ X_{\Sigma_0}. & & \end{array}$$

Note that $X_{\Sigma'}$ is simplicial and projective with $|\Sigma'| = |\Sigma|$. Replace X_Σ and D with $X_{\Sigma'}$ and the birational transform $\psi_* D$. Return to step (a) and continue.

The procedure terminates with a composition

$$\psi : X_\Sigma \dashrightarrow X_{\Sigma'} \dashrightarrow \cdots \dashrightarrow X_{\Sigma_*}$$

of D-negative divisorial extremal contractions and elementary flips such that either

1. There is a D-negative fibering contraction from X_{Σ_*} to a toric variety of smaller dimension, or

2. $\psi_* D$ is nef on X_{Σ_*}.

Proposition 3.2 *[CLS11] Let D be a Weil divisor on a simplicial projective toric variety X_Σ. Then the D-negative extremal rays in Proposition 3.1 for X_Σ and D can be chosen so that the procedure stops after finitely many iterations.*

Now let's use the combinatorial description in terms of the secondary fan associated with the set of rays in the fan of X to analyze the sequence of birational modifications $X \dashrightarrow Y \to Z$.

Because the toric varieties X and Z have the same rays, they are isomorphic in codimension one, and we have $\mathrm{Cl}(X) \simeq \mathrm{Cl}(Z)$. We see $-K_X = -K_Z$ under this identification.

Consider the secondary fan of X in $\mathrm{Pic}(X)_{\mathbb{R}}$, we know $\mathrm{Nef}(X)$ is a chamber in the secondary fan. Also, since Z is projective, $-K_Z$ is ample, Σ_Z and Σ_X have the same rays, we know $\mathrm{Nef}(Z)$ is a possibly lower dimension cone in the secondary fan of X. Choose a chamber $\mathrm{Nef}(Y)$, which contains $\mathrm{Nef}(Z)$ as a face.

Let $f : Y \dashrightarrow Z$ be the natrual birational map. Because Y and Z are isomorphic up to codimension two, we have $-K_Y = f^*(-K_Z)$. So $-K_Y$ is nef, since $-K_Z$ is ample and $\mathrm{Nef}(Z) \subset \mathrm{Nef}(Y)$.

Now we can draw a line segment between a generic point in the interior of $\mathrm{Nef}(X)$ to $-K_X = -K_Z$ in the interior of $\mathrm{Nef}(Z)$ such that the segment always crosses from one chamber to another at a relative interior point of a wall. This is possible since the support of the secondary fan of X is equal to the Moving cone of X, which is convex by ([CLS11], Proposition 15.1.4.) .

Consider the wall Γ which the line segment crosses first, this gives a curve $C_1 \subset X$ such that $-K_X \cdot C_1 < 0$ and C_1 generates an edge of the Mori cone of X, and an extremal contraction $\phi_1 : X \to X_1$. This wall can be a divisorial wall or a flipping wall by the definition of a wall, which corresponds to a divisorial contraction or a flip. But in our case, since we know the final step of the sequence of maps is Z and $X \dashrightarrow Z$ is isomorphic up to codimension two, the wallcrossing map in the middle steps can only be a flip map.

If $\phi_{1*}(-K_X)$ is nef on X_1, then we are done and $X_1 = Y$. If $\phi_{1*}(-K_X)$ is not nef on X_1, then the line segment will cross the next wall, which is a facet of $\mathrm{Nef}(X_2)$, this gives a curve $C_2 \subset X_1$ such that $-K_{X_1} \cdot C_2 < 0$, and a flip $\phi_2 : X_1 \to X_2$. Continue these steps, we will finally get $X_n = Y$, and a flip $\phi_{n-1} : X_{n-1} \to X_n = Y$, such that $\phi_{n-1*}(-K_{X_{n-1}}) = -K_Y$ is nef on Y. This procedure terminates in finitely many steps since the line segment meets only finitely many chambers. (There are finitely many chambers in the secondary fan of a projective toric variety).

Now we consider the map $Y \to Z$. Since Y is a toric variety and $D := -K_Y$ is a nef divisor, we know nD is basepoint free for some $n > 0$, and we have a morphism

$$\varphi : Y \underset{|nD|}{\longrightarrow} Z,$$

where φ contracts all the curves $C \subset Y$ such that $-K \cdot C = 0$. That's the last part of our line segment which connects a point in the interior of $\mathrm{Nef}(Y)$ and $-K_X = -K_Z$ in the interior of $\mathrm{Nef}(Z)$.

3.2 Compactification of cluster variety

Now, we will have our main theorem in this section. But before going to the main theorem of this section, let's review some definitions of singularities of a variety.

Definition 3.3 *We say a variety X is* $\mathbb{Q}$-factorial *if every divisor D on X is a $\mathbb{Q}$-Cartier divisor, i.e., there exists $n \in \mathbb{N}$ such that nD is a Cartier divisor.*

The next lemma is an equivalent definition of a $\mathbb{Q}$-factorial toric variety.

Lemma 3.4 *([CLS11], Proposition 4.2.7 and Theorem 11.4.8)*

Let Y be a toric variety, then Y is $\mathbb{Q}$-factorial if and only if the fan of Y is simplicial, i.e., each cone σ of the fan is generated by $\dim \sigma$ rays. Equivalently Y has only (abelian) quotient singularities.

Definition 3.5 *Suppose that X is a complex normal variety such that its canonical divisor K_X is $\mathbb{Q}$-Cartier. Let $f : Y \longrightarrow X$ be a resolution of the singularities of X such that the exceptional divisor is simple and normal crossing. Then*

$$K_Y = f^*(K_X) + \sum_i a_i E_i$$

where the sum is over the irreducible exceptional divisors E_i. The rational numbers a_i are called discrepancies*. We write $a(E_i, X) := a_i$.*

Then we say that the singularities of X are

- terminal *if $a_i > 0$ for all i,*
- canonical *if $a_i \geq 0$ for all i,*
- log terminal *if $a_i > -1$ for all i,*
- log canonical *if $a_i \geq -1$ for all i.*

We can also consider the discrepancy of pairs (X, D).

Definition 3.6 *Let (X, D) be a pair where X is a complex normal variety and D is an divisor on X such that $K_X + D$ is $\mathbb{Q}$-Cartier. Then a* log resolution *of (X, D) is a proper and birational morphism $f : Y \longrightarrow X$ such that $Exc(f) \cup f_*^{-1}D$ has simple normal crossing support, where $Exc(f)$ is the* exceptional locus *of the map f, and $f_*^{-1}D$ is the strict transform of D on Y. Then, we can write*

$$K_Y + D' = f^*(K_X + D) + \sum_{E_i:exceptional} a(E_i, X, D)E_i.$$

The rational number $a(E, X, D)$ is called the discrepancy *of E with respect to (X, D). The* discrepancy *of (X, D) is given by*

$$discrep(X, D) = \inf_E \{a(E, X, D) | E \text{ is an exceptional divisor over } X\}.$$

Then we say that (X, D) is log canonical *if $discrep(X, D) \geq -1$.*

We will use a lemma from [KM] to prove the relation of log canonical between two pairs.

Lemma 3.7 ([KM], Lemma 2.30) *Let $f : Y \longrightarrow X$ be a proper birational morphism between normal varieties. Let D_Y be a $\mathbb{Q}$-divisor on Y and D_X be a $\mathbb{Q}$-divisor on X such that*

$$K_Y + D_Y \equiv f^*(K_X + D_X) \quad \textit{and} \quad f_* D_Y = D_X.$$

Then, for any divisor F over X,

$$a(F, Y, D_Y) = a(F, X, D_X)$$

Now we can prove the following lemma about the log canonical property of two pairs.

Lemma 3.8 *Let X and Y be normal varieties. $h : X \dashrightarrow Y$ is an isomorphism up to codimension two. Let D_Y be a $\mathbb{Q}$-divisor on Y and D_X be a $\mathbb{Q}$-divisor on X such that*

$$K_Y + D_Y = 0 \quad \textit{and} \quad K_X + D_X = 0.$$

Then if (X, D_X) log canonical, (Y, D_Y) is also log canonical.

Proof. We can choose a log resolution $f : Z \longrightarrow X$ of (X, D_X), write

$$K_Z + D'_X = f^*(K_X + D_X) + \sum_i a_i E_i,$$

and define

$$D_Z = D'_X - \sum_i a_i E_i.$$

Then we have

$$f_* D_Z = D_X \text{ and } K_Z + D_Z = f^*(K_X + D_X) = 0.$$

We may assume the birational map have $g : Z \longrightarrow Y$ is a morphism. Since $h : X \dashrightarrow Y$ is an isomorphism up to codimension two, we also have

$$K_Z + D_Z = f^*(K_Y + D_Y) = 0 \quad \text{and} \quad g_* D_Z = D_Y.$$

Now, we use Lemma 3.7, we know for any divisor F over X,

$$a(F, Z, D_Z) = a(F, X, D_X)$$

and for any divisor F over Y

$$a(F, Z, D_Z) = a(F, Y, D_Y).$$

Thus

$$\mathrm{discrep}(X, D_X) = \mathrm{discrep}(Y, D_Y).$$

This means if (X, D_X) log canonical, then (Y, D_Y) is also log canonical.

$\diamond$

We also have the following lemma about the log terminal property of a toric variety.

Lemma 3.9 *Let Z be a toric variety. Assume K_Z is $\mathbb{Q}$-Cartier, then Z has log terminal singularities.*

Proof.

Let $B_Z \subset Z$ be the toric boundary, then (Z, B_Z) is log canonical. So $\mathrm{Sing}(Z) \subset B_Z$.

Assume K_Z is $\mathbb{Q}$-Cartier, so $B_Z = -K_Z$ is also $\mathbb{Q}$-Cartier.

Let the map $\pi : \tilde{Z} \to Z$ be the log resolution of Z. It is an isomophism over $Z \backslash B_Z$. Then

$$(K_{\tilde{Z}} + B'_Z) = \pi^*(K_Z + B_Z) + \sum a_i(Z, B_Z)E_i.$$

Since (Z, B_Z) is log canonical, we have $a_i(Z, B_Z) \geq -1$. Also we have $B'_Z = \pi^* B_Z - \sum \mu_i E_i$, $\mu_i \in \mathbb{Q}$, $\mu_i \geq 0$.

Thus

$$K_{\tilde{Z}} = \pi^* K_Z + \pi^* B_Z + \sum a_i(Z, B_Z) E_i - (\pi^* B_Z - \sum \mu_i E_i).$$
$$= \pi^* K_Z + \sum (a_i(Z, B_Z) + \mu_i) E_i$$

We may assume $\pi(E_i) \subset \mathrm{Sing}(Z) \subset B_Z$, then we have $\mu_i > 0$.

Thus $a_i(Z) = a_i(Z, B_Z) + \mu_i > -1$ since $\mu_i > 0$ for all i and $a_i(Z, B_Z) \geq -1$.

So Z has log terminal singularities.

$\diamond$

The next theorem is the main theorem of this section, which gives a nice compactification of the $\mathcal{X}$-cluster variety of finite type for type A, D, E root system Φ up to codimension two.

Theorem 3.10 *Let U be a $\mathcal{X}$-cluster variety of finite type for the A, D, E root system Φ. Then there exists a compactification $U = Z \setminus D_Z$ up to codimension two such that the following conditions hold:*

1. *Z is a toric variety with log terminal singularities and $-K_Z$ is ample.*

2. *$D_Z = H_{1,Z} \cup ... \cup H_{n,Z} \bigcup B_{1,Z} \cup ... \cup B_{n,Z}$, $H_{i,Z}$ are the corresponding hypertori $H_{i,Z} = \overline{(\mathcal{X}^{e_i} = 1)}$ in Z, and $B_{i,Z}$ are the toric boundary divisors on Z corresponding to the ray $\mathbb{R}_{\geq 0} \cdot (-f_i)$,$i = 1, ..., n$.*

3. *$K_Z + D_Z = 0$. In particular, D_Z is ample and U_Z is affine.*

4. *(Z, D_Z) is log canonical.*

Proof.

From Theorem 2.8, we have the following map,

$$((\mathbb{P}^1)^n, \bar{B}) \overset{n \text{ blowups}}{\longleftarrow} (X, D),$$

where $\bar{B}$ is the toric boundary of $(\mathbb{P}^1)^n$, and $X \setminus D$ is isomorphic to the $\mathcal{X}$-cluster variety of finite type for type A, D, E root system Φ up to codimension two.

Let $D = \sum_{i=1}^n H_i + \sum_{i=1}^n D_i$, where H_i are the hypertori, and D_i are the toric boundary divisors. We have $K_X + D = 0$ by Lemma 2.10.

X is the toric variety defined in Theorem 2.8, which corresponds to the root system Φ. X has rays $\rho_i = \mathbb{R}_{\geq 0} \cdot v_i$ for primitive generators $v_i \in N$, $i = 1, 2, ..., 3n$, where $v_i = f_i$, $v_{n+i} = f_i - \sum_{k \neq i} |b_{ik}| f_k$, $v_{2n+i} = -f_i$, $i = 1, ..., n$. Then $P := Conv(v_1, ...v_{3n}) \subset N_{\mathbb{R}}$ is a Convex polytope containing 0 in its interior and with vertices $S = \{v_1, ...v_{3n}\}$ by Lemma 2.16.

Let Z be the toric variety corresponding to the polytope P defined in Lemma 2.17. Then Z is a toric Fano variety by Lemma 2.17.

Since the fans correspond to toric varieties X and Z have the same rays, we have a birational map $(X, D) \dashrightarrow (Z, D_Z)$, which is an isomorphism in codimension one.

By the previous discussion of this section, we can decompose the map with two maps

$$\begin{array}{ccccc} (X, D) & \dashrightarrow & (Y, D_Y) & \longrightarrow & (Z, D_Z) \\ \downarrow & & & & \\ ((\mathbb{P}^1)^n, \bar{B}) & & & & \end{array}$$

The first map is the D-minimal model program, for $D = -K_X$. We contract curves C such that $D \cdot C < 0$ and do the D-flips($= K_X$ anti-flips). The second map contracts curves C s.t. $D \cdot C = 0$. These maps are toric isomorphisms in codimension one, because the fans of the toric varieties X, Y, Z all have the same rays. But D is not the toric boundary of X.

We have $-K_Y$ is nef by the D-minimal model program, and $-K_Z$ is ample by the construction of Z.

For (X, D), it is smooth and normal crossing, since it is a blowup of $(\mathbb{P}^1)^n$, so it is divisorial log terminal and also log canonical.

Since $K_X + D = 0$, we know

$$K_Y + D_Y = 0 \quad \text{and} \quad K_Z + D_Z = 0.$$

This is because X, Y, Z are isomorphic in codimension one.

Thus (Y, D_Y) and (Z, D_Z) are also log canonical by Lemma 3.8, and $U \simeq Z \setminus D_Z$ up to codimension two.

D_Z is also ample since $D_Z \sim -K_Z$. Now we can prove $U_Z = Z \setminus D_Z$ is affine.

Because Z is a projective toric variety and D_Z is ample, there exists $n \in \mathbb{N}$ such that nD_Z is very ample, i.e., $nD_Z = Z \cap H$, for some embedding

$$Z \underset{H^0(\mathcal{O}_Z(nD_Z))}{\hookrightarrow} \mathbb{P}^N,$$

where $H \subset \mathbb{P}^N$ is a hyperplane. So nD_Z is a hyperplane section.

Now we have

$$U_Z = Z \setminus D_Z = Z \setminus nD_Z = Z \cap (\mathbb{P}^N \setminus H) = Z \cap \mathbb{C}^N,$$

which is a closed subset of the affine variety $\mathbb{C}^N$. Thus $U_Z = Z \setminus D_Z$ is affine.

Moreover, since the maps are isomorphism in codimension one, they have the same divisors. We have

$$D_Z = \sum_{i=1}^{n} H_{i,Z} + \sum_{i=1}^{n} B_{i,Z},$$

where $H_{i,Z}$ are the corresponding hypertori $H_{i,Z} = \overline{(\mathcal{X}^{e_i} = 1)}$ in Z, and $B_{i,Z}$ are the toric boundary divisors on Z corresponding to the rays $\mathbb{R}_{\geq 0} \cdot (-f_i)$,$i = 1, ..., n$.

Since K_Z is $\mathbb{Q}$-Cartier, so by lemma 3.9, we know Z has log terminal singularities.

◇

Remark 3.11 *In fact, Z has terminal singularities, see Theorem 3.22.*

Lemma 3.12 *Let X, Y and Z be the toric varieties defined in Theorem 3.10. Then X and Y are $\mathbb{Q}$-factorial varieties, and Z is not $\mathbb{Q}$-factorial in general.*

Proof.

Since X is a blowup of $(\mathbb{P}^1)^n$, it is a smooth variety. Thus X is $\mathbb{Q}$-factorial.

Because the sequence of maps from X to Y is the D-minimal model program, Y is also $\mathbb{Q}$-factorial. And since Y is toric, equivalently, we know Y has abelian quotient singularities or equivalently the fan of the toric variety Y is simplicial.

However, in general, Z won't be $\mathbb{Q}$-factorial.

◇

We have some examples for which we can analyze the maps

$$(X, D) \dashrightarrow (Y, D_Y) \longrightarrow (Z, D_Z)$$

explicitly.

Example 3 *For the root system A_3, the map $(X, D) \dashrightarrow (Y, D_Y)$ is an isomorphism.*

This is because the map "flips" the curves $C \subset X$ with $(-K) \cdot C < 0$, i.e., $K \cdot C > 0$, but for any curves $C \subset X$, $K \cdot C \leq 0$. This can be checked by PARI/GP, see the appendix for details.

So no curves will flip, in other words, $-K_X$ is already nef, so by definition of minimal model program, $X = Y$. This means the map is an isomorphism.

Example 4 *We have an example such that the map $(X, D) \dashrightarrow (Y, D_Y)$ is not an isomorphism.*

By PARI/GP (see the appendix for details) we know that for A_4, there exists some curve $C \subset X$ with $(-K)\cdot C < 0$, so we need to flip this curve by construction. So the map $X \dashrightarrow Y$ is not an isomorphism in general.

Example 5 *B_2, G_2. Unlike the A, D, E cases, where the maps $X \dashrightarrow Y \to Z$ are isomorphisms up to codimension two (which means they are isomorphisms in surface case), for B_2, G_2 cases, the maps are not isomorphisms. But the following statements are still true.*

There exists a compactification $U = Z \setminus B_Z$ up to codimension two such that the following conditions hold:

(1). Z is a toric variety with log terminal singularities and $-K_Z$ is ample.

(2). $K_Z + B_Z = 0$. In particular, B_Z is ample and U is affine. Moreover (Z, B_Z) is log canonical.

B_2

$$(X, B) \xrightarrow{\sim} (Y, B_Y) \xrightarrow{f_{B_2}} (Z, B_Z)$$

Here f_{B_2} contracts (-2) curves.

G_2.

$$(X, B) \xrightarrow[f_{G_2}]{} (Y, B_Y) \xrightarrow{\sim} (Z, B_Z)$$

Here f_{G_2} contracts (-3) curves.

Remark 3.13 *Now $Y = Z$ for G_2, and Z is not Gorenstein.*

3.3 The polytope P

We will analyze the properties of the toric variety Z by analyzing the corresponding polytope P. We first review some definitions and correspondence between the variety and the polytope.

Definition 3.14 *Let X be a normal complex variety and D be a Weil divisor, i.e., a combination of codimension one subvarieties. We say that D is* $\mathbb{Q}$-Cartier *if there exists an integer m such that mD is Cartier. The smallest such m is called the* index m_D *of D.*

Let K_X be the canonical divisor *of X, i.e., a Weil divisor of X whose restriction to the regular locus defines the canonical sheaf there.*

A complex variety X is called Gorenstein, *if $m_{K_X} = 1$, i.e., K_X is a Cartier divisor.*

A variety X is called Fano *variety if X is projective, normal and the anticanonical divisor $-K_X$ is an ample $\mathbb{Q}$-Cartier divisor.*

Definition 3.15 *Let $P \subset N_{\mathbb{R}}$ be a d-dimensional lattice polytope with $0 \in int\ P$, and let $V(P)$ be the vertex set of the polytope P. Then*

1) P is called a Fano *polytope if each vertex $v \in V(P)$ is a primitive lattice point of N.*

2) P is called a canonical Fano *polytope, if $int\ P \cap N = \{0\}$.*

3) P is called a terminal Fano *polytope, if $P \cap N = \{0\} \cup V(P)$.*

4) P is called a smooth Fano *polytope, if the vertices of any facet of P form a $\mathbb{Z}$-basis of the lattice N.*

Lemma 3.16 *[BN] Let X be the toric variety corresponding to the polytope P, i.e., the fan of X is the face fan of P. Then*

1) If P is a Fano polytope, then X is Fano.

2) If P is a canonical Fano polytope, then X has canonical singularities.

3) If P is a terminal Fano polytope, then X has terminal singularities.

4) If P is a smooth Fano polytope, then X is a smooth Fano variety.

Lemma 3.17 *[BN] Let X be a compact toric variety with fan Σ in $\mathrm{N}_{\mathbb{R}}$. Then the following statements are equivalent:*

1) X has terminal singularities.

2) For all maximal cones $\sigma \in \Sigma$, let $v_1, ..., v_r$ be the primitive generators of the rays of σ, then we have $Conv(0, v_1, ..., v_r) \cap N = \{0, v_1, ..., v_r\}$.

Definition 3.18 *A Fano polytope $P \subset N_{\mathbb{R}}$ is called* reflexive *if the* dual polytope *$P^* := \{u \in M_{\mathbb{R}} \mid \langle u, v\rangle \geq -1 \text{ for all } v \in P\}$ is a lattice polytope. Equivalently, each facet of P has lattice distance one from 0.*

Lemma 3.19 *[BN] Let $P \subset N_{\mathbb{R}}$ be a d-dimensional lattice polytope with $0 \in int\, P$. Then the following conditions are equivalent:*

1) P is a reflexive polytope

2) P is a lattice polytope and P^* is a lattice polytope

3) P^* is a reflexive polytope

If this holds, then $int\, P \cap N = \{0\}$, i.e., P is a canonical Fano polytope.

$\mathbb{Q}$-factorial toric Fano varieties correspond uniquely up to isomorphism to simplicial Fano polytopes.

Gorenstein toric Fano varieties correspond uniquely up to isomorphism to reflexive polytopes. This is shown by the following lemma.

Lemma 3.20 *[CLS11] Let X be a normal toric variety. If X is a projective Gorenstein Fano variety, then the polytope associated to the anti-canonical divisor $-K_X = \sum_\alpha D_\alpha$ is reflexive. Conversely, if X_P is the projective toric variety associated to a reflexive polytope P, then X_P is a Gorenstein Fano variety.*

Actually, the toric variety Z has terminal singularities if we can prove the corresponding polytope P is terminal.

Lemma 3.21 *Let $P := Conv(v_1, ..., v_{3n}) \subset N_{\mathbb{R}}$ be the convex polytope defined in Lemma 2.16. Then 0 is a unique lattice point contained in the interior of P, moreover, $P \cap N = \{0, v_1, ..., v_{3n}\}$, i.e., P is terminal.*

Proof. See the appendix. $\diamond$

Theorem 3.22 *Let Z be the toric variety corresponding to the face fan of P, then Z has terminal singularities, and Y is also terminal.*

Proof. By Lemma 3.21, the corresponding polytope of Z is terminal, so Z has terminal singularities.

Let $f : \tilde{Y} \longrightarrow Z$ be a resolution of the singularities of Z. Then

$$K_{\tilde{Y}} = f^*(K_Z) + \sum_i a_i E_i$$

where the sum is over the irreducible exceptional divisors E_i of f. Since Z has terminal singularities, we know $a_i > 0$ for all i.

Let $g : Y \longrightarrow Z$ be the morphism constructed in Theorem 3.10. We may assume that f factors through Y, i.e., we have a commutative diagram:

$$\begin{array}{ccc} \tilde{Y} & & \\ \downarrow{\scriptstyle h} & \searrow^{f} & \\ Y & \xrightarrow{g} & Z \end{array}$$

Since g is an isomorphism up to codimension two, so g has no exceptional divisors, we know exceptional divisors of $f = g \circ h$ are exceptional divisors of h.

Now we have

$$\begin{aligned} K_{\tilde{Y}} &= f^*(K_Z) + \sum_i a_i E_i \\ &= h^* g^*(K_Z) + \sum_i a_i E_i \\ &= h^*(K_Y) + \sum_i a_i E_i \end{aligned}$$

Since $a_i > 0$ for all i, Y has terminal singularities. ◇

Now we have the question: is the anti-canonical divisor $-K_Y$ Cartier?

Because $f : Y \longrightarrow Z$ is a morphism, we know $-K_Y = f^*(-K_Z)$. So if $-K_Z$ is a Cartier divisor, $-K_Y$ is also Cartier.

Since Z is a toric Fano variety which corresponds to the polytope

$$P := Conv(v_1, ...v_{3n}) \subset N_{\mathbb{R}},$$

then $-K_Z$ Cartier if only if P is a reflexive polytope by Lemma 3.20.

Theorem 3.23 *P is a reflexive polytope for A_n and D_n when $n \leq 19$, and E_6, E_7, E_8. Then K_Z is a Cartier divisor, and Z is Gorenstein. This means K_Y is also Cartier for these cases.*

Proof. We can check that P is a reflexive for A_n and D_n when $n \leq 19$, and E_6, E_7, E_8 using the program [PALP].

Then K_Z is a Cartier divisor. Since $f : Y \longrightarrow Z$ is a morphism and an isomorphism up to codimension 2, we know that $K_Y = f^* K_Z$. So K_Y is also a Cartier divisor. ◇

Now we have the following conjecture.

Conjecture 3.24 *P is a reflexive polytope for all type A, D, E root systems, so Z is a Gorenstein Fano toric variety for all type A, D, E root systems.*

Example 6 *Since we have the exact toric information of the toric variety Z, we can analyze Z explicitly for A_3, A_4, D_4 by computing the facets of the polytope P and studying the associated singularities. We will compute it for A_3 explicitly. For the cases A_4 and D_4 it will be very similar.*

1. For A_3, we have 10 facets of P corresponding to 4 ordinary double points and 6 smooth points. Z is a Fano 3-fold with four ordinary double points.

   P has 9 vertices, which are $\pm e_1, \pm e_2, \pm e_3, e_1 - e_2, -e_1 + e_2 - e_3, -e_2 + e_3$. Using [PALP], we can compute the vertices of the dual polytope P^* explicitly, which correspond to the facets of the polytope P.

   P^* has 10 vertices.

   For every vertex, we can compute the facet of P.

   The vertex $(-1, -1, 1) \in N$ corresponds to the supporting hyperplane $-x_1 - x_2 + x_3 = -1$ of P, so the corresponding facet is spanned by $e_1, e_2, -e_3, -e_1 + e_2 - e_3$. We have $e_2 + -e_3 = -e_1 + e_2 - e_3 + e_1$, which implies that the corresponding point is an ordinary double point $(xy = zw) \subset \mathbb{A}^4$.

   The vertex $(1, 1, 0) \in N$ corresponds to the supporting hyperplane $x_1 + x_2 = -1$ of P, so the corresponding facet is spanned by $-e_1, -e_2, -e_2 + e_3$, it is a smooth point.

   For the rest of the vertices, we can do similar calculations, and we find that there are 4 ordinary double points and 6 smooth points in total.

2. For A_4, we have 23 facets of P, corresponding to 13 smooth points, 9 points

of type $TV(\sigma) = \mathbb{A}^1 \times (3 \text{ fold } ODP)$ and 1 special point. The special point is of type $(x_1x_2x_3 = y_1y_2) \subset \mathbb{A}^5$.

See appendix 1 for the vertices of the dual polytope of A_4.

3. For D_4, we have 24 facets of P corresponding to 14 smooth points, 9 points of type $TV(\sigma) = \mathbb{A}^1 \times (3 \text{ fold } ODP)$, and 1 special point. The special point is of type $(x_1x_2x_3 = y_1y_2) \subset \mathbb{A}^5$.

   See appendix 2 for the vertices of the dual polytope of D_4.

CHAPTER 4

RELATION TO TORIC VARIETY FOR FAN OF WEYL CHAMBERS

In this chapter, we will consider the toric variety associated with the root system and relate it to the cluster variety for the root system. Toric varieties associated with the root systems are well-known, we give more details to set up the notations.

For a root system Φ of rank n, we can get a n-dimensional smooth projective toric variety $X(\Phi)$ associated with its fan of Weyl chambers.

Let Φ be a root system of rank n in the n dimensional space E, M be the root lattice, and let N be the lattice dual to M. Let $\Delta = \{\alpha_1, ...\alpha_n\}$ be the set of simple roots. It is a basis of M, and $\Phi = \Phi^+ \cup \Phi^- \subset M$, $\Phi^- = -\Phi^+$. $\Phi^+ \subset \{\alpha = \sum m_i \alpha_i \mid m_i \geq 0, m_i \in \mathbb{Z}\}$.

Let

$$W(\Phi) = \langle s_\alpha \mid \alpha \in \Phi \rangle = \langle s_\alpha \mid \alpha \in \Delta \rangle$$

be the Weyl group of the root system Φ, where s_α is the reflection about the hyperplane perpendicular to α, given by

$$s_\alpha(m) = m - 2\frac{(\alpha, m)}{(\alpha, \alpha)}\alpha$$

for $m \in M$, where $(\cdot, \cdot) : M \times M \longrightarrow \mathbb{Z}$ is a positive definite symmetric bilinear form defined by

$$(\alpha_i, \alpha_j) = \begin{cases} 2 & \text{if } i = j \\ -|b_{ij}| & \text{if } i \neq j \end{cases}$$

For every root α, we have the hyperplane perpendicular to the root α, denoted by $\alpha^\perp$, and $\alpha^\perp \subset N_\mathbb{R}$ is a reflection hyperplane. The complement of the set of hyperplanes is disconnected, and these reflection hyperplanes $\{\alpha^\perp \mid \alpha \in \Phi\}$ subdivide $N_\mathbb{R}$ into strictly simplicial cones. Each connected component is called a Weyl chamber.

Let $\Sigma_{X(\Phi)}$ be the fan consisting of the Weyl chambers and all their faces. These Weyl chambers cover $N_\mathbb{R}$, so the fan $\Sigma_{X(\Phi)}$ is complete. Each chamber C corresponds to basis of simple roots $\alpha_1, \alpha_2, ..., \alpha_n$ via $C = (\alpha_1, \alpha_2, ..., \alpha_n \geq 0) \subset N_\mathbb{R}$. ($C$ is generated by the dual basis $\alpha_1^*, \alpha_2^*, ..., \alpha_n^*$ of N).

Let $X(\Phi)$ be the toric variety associated with the fan $\Sigma_{X(\Phi)}$, it is called the toric variety associated with the root system Φ. Then $X(\Phi)$ is a compact smooth toric variety.

$W = W(\Phi)$ acts simply transitively on the set of Weyl Chambers. So W acts on $X(\Phi)$ and permutes the zero-dimensional orbits (fixed points of the torus action) simply transitively.

The boundary divisors of $X(\Phi)$ correspond to the rays of the fan Σ.

Let Φ be a type A, D, E root system. Now we have two varieties related to the root system Φ: the variety X defined in Theorem 2.8, which is the blowup of n loci in $(\mathbb{P}^1)^n$; and the variety $X(\Phi)$, which is the toric variety $X(\Phi)$ associated with the fan of Weyl chambers. Let Σ_X be the fan of X, and $\Sigma_{X(\Phi)}$ be the fan of $X(\Phi)$. We observe the following connection between the $\mathcal{X}$ cluster variety for Φ and toric variety $X(\Phi)$ for the fan of Weyl chambers.

Theorem 4.1 *The rays of the fan Σ_X form a subset of the rays of $\Sigma_{X(\Phi)}$, so that X is a toric open set in $X(\Phi)$ up to codimension two. Equivalently, writing $f : X(\Phi) \dashrightarrow X$ for the birational map given by the identification of tori, the inverse birational map f^{-1} has no exceptional divisors.*

Proof.

To prove X is a toric open set in $X(\Phi)$ up to codimension two, we only need to show the rays of Σ_X form a subset of the rays of $\Sigma_{X(\Phi)}$. By Theorem 2.8, we know Σ_X has rays $\{\rho_1, \rho_2, ..., \rho_{3n}\}$, where

$$\rho_i = \mathbb{R}_{\geq 0} \cdot f_i, \quad \rho_{n+i} = \mathbb{R}_{\geq 0} \cdot (f_i - \sum_{k=1, k\neq i}^{n} |b_{ik}| \cdot f_k), \quad \rho_{2n+i} = \mathbb{R}_{\geq 0} \cdot (-f_i), i = 1, 2, ..., n.$$

For type A, D, E root systems,

$$b_{ij} = \begin{cases} 1 & \text{if } ij \text{ is an edge} \\ 0 & \text{otherwise} \end{cases}$$

And $f_1, f_2, ..., f_n$ is the dual basis to $\alpha_1, \alpha_2, ..., \alpha_n$, which is a basis of simple roots of root lattice.

Then $\rho_i = \mathbb{R}_{\geq 0} \cdot f_i$, $i = 1, 2, ..., n$, are rays of the maximal cone $C = (\alpha_1, ..., \alpha_n \geq 0)$, C is the fundamental chamber.

Thus $C = \langle f_1, f_2, ..., f_n \rangle_{\mathbb{R}_{\geq 0}} \subset \Sigma_{X(\Phi)}$, so we have $\rho_i = \mathbb{R}_{\geq 0} \cdot f_i \in \Sigma_{X(\Phi)}$, $i = 1, 2, ..., n$.

Since $\Sigma_{X(\Phi)}$ is determined by hyperplanes, we have $\Sigma_{X(\Phi)} = -\Sigma_{X(\Phi)}$.

Thus $\rho_{2n+i} = \mathbb{R}_{\geq 0} \cdot (-f_i) \in \Sigma_{X(\Phi)}$, $i = 1, 2, ..., n$.

Finally, let's show

$$\rho_{n+i} = \mathbb{R}_{\geq 0} \cdot (f_i - \sum_{k=1, k\neq i}^{n} |b_{ik}| \cdot f_k) \in \Sigma_{X(\Phi)}, \;\; i = 1, 2, ..., n.$$

Since $(\alpha, \alpha) = 2$ for any root α, we have

$$s_\alpha(m) = m - 2\frac{(\alpha, m)}{(\alpha, \alpha)}\alpha = m - (\alpha, m)\alpha$$

for $m \in M$.

We have an isomorphism

$$\psi : M_{\mathbb{Q}} \longrightarrow N_{\mathbb{Q}}, \quad \alpha \mapsto (\alpha, \cdot).$$

We define the corresponding Weyl group on N, $S_\alpha : N \longrightarrow N$ by

$$S_\alpha(v) = S_{\psi(\alpha)}(v) = v - (\psi(\alpha), v)\psi(\alpha) = v - (\psi(\alpha), v)(\alpha, \cdot)$$

for $v \in N$, where $(\cdot, \cdot) : N_{\mathbb{Q}} \times N_{\mathbb{Q}} \longrightarrow \mathbb{Z}$ is the form on $N_{\mathbb{Q}}$ induced by the form $(\cdot, \cdot)$ on M via ψ; that is

$$(\psi(\alpha), \psi(\beta)) = (\alpha, \beta) \text{ for } \alpha, \beta \in M,$$

or equivalently,

$$(\psi(\alpha), v) = \langle \alpha, v \rangle \text{ for } \alpha \in M, v \in N.$$

S_α is a reflection in $\alpha^\perp$ with respect to inner product on $N = M^*$ determined by $(\cdot, \cdot)$.

Since

$$(\alpha_i, \alpha_j) = \begin{cases} 2 & \text{if } i = j \\ -|b_{ij}| & \text{if } i \neq j \end{cases}$$

we have

$$(\alpha_i, \cdot) = \sum_{j=1}^{n} (\alpha_i, \alpha_j)\alpha_j^* = 2\alpha_i^* - \sum_{i \neq j} |b_{ij}|\alpha_j^*.$$

Then

$$\begin{aligned}
S_{\alpha_i}(f_i) &= f_i - (\psi(\alpha_i), f_i) \cdot (\alpha_i, \cdot) \\
&= f_i - 1 \cdot (2\alpha_i^* - \sum_{i \neq j} |b_{ij}| \alpha_j^*) \\
&= f_i - (2f_i - \sum_{i \neq j} |b_{ij}| f_j) \\
&= -(f_i - \sum_{i \neq j} |b_{ij}| f_j)
\end{aligned}$$

so we have

$$S_{\alpha_i}(-f_i) = (f_i - \sum_{i \neq j} |b_{ij}| f_j).$$

Because

$$\rho_{2n+i} = \mathbb{R}_{\geq 0} \cdot (-f_i) \in \Sigma_{X(\Phi)},$$

thus we know

$$\rho_{n+i} = \mathbb{R}_{\geq 0} \cdot (f_i - \sum_{k=1, k \neq i}^{n} |b_{ik}| \cdot f_k) \in \Sigma_{X(\Phi)}, \ i = 1, 2, ..., n.$$

So we have shown that all the rays of Σ_X are a subset of rays of $\Sigma_{X(\Phi)}$.

Thus X is a toric open set in $X(\Phi)$ up to codimension two. Equivalently, writing $f : X(\Phi) \dashrightarrow X$ for the birational map given by the identification of tori, the inverse birational map f^{-1} has no exceptional divisors.

◇

Now for any root system Φ, we consider the polytope $\tilde{P}$ corresponding to the toric variety $X(\Phi)$.

Let

$$\tilde{P} = Conv\{v \mid \rho = \mathbb{R}_{\geq 0} \cdot v \text{ a ray in } \Sigma_{X(\Phi)}, v \in N \text{ the primitive generator of } \rho\},$$

then $W = W(\Phi)$ acts on $\tilde{P}$. The polytope P defined in Lemma 3.21 is a subset of $\tilde{P}$ by Theorem 4.1, since $\{v_1, v_2, ..., v_{3n}\}$ is a subset of the set of the primitive generators of rays of fan $\Sigma_{X(\Phi)}$ of Weyl chambers of the root system Φ.

Question 4.2 *Now consider the following two questions:*

1. *Is each v a vertex of $\tilde{P}$?*

2. *Is $\tilde{P}$ a terminal polytope? i.e.,*

$$\tilde{P} \cap N = \{0\} \cup \{v \mid \rho = \mathbb{R}_{\geq 0} \cdot v \textit{ a ray in } \Sigma_{X(\Phi)}, v \in N \textit{ the primitive generator of } \rho\}$$

Let $\tilde{X}$ be the toric variety corresponding to polytope $\tilde{P}$, where $\Sigma_{\tilde{X}}$ is the spanning fan of $\tilde{P}$. We will prove both (1) and (2) are true when $\Phi = A_n$, moreover $\tilde{P}$ is a reflexive polytope when $\Phi = A_n$, so $\tilde{X}$ is a terminal Gorenstein toric Fano variety by Lemma 2.17.

Also, we will prove if (2) is true for $\tilde{P}$, then (2) is true for P. This gives another proof that P is a terminal polytope when $\Phi = A_n$.

Lemma 4.3 *Let $\Phi = A_n$ be a root system. Let*

$$\tilde{P} = Conv\{v \mid \rho = \mathbb{R}_{\geq 0} \cdot v \textit{ a ray in } \Sigma_{X(\Phi)}, v \in N \textit{ the primitive generators of } \rho\}.$$

Then each v is a vertex of $\tilde{P}$, and $\tilde{P}$ is a reflexive and terminal polytope. Thus $\tilde{X}$ is a terminal Gorenstein toric Fano variety.

Proof. For the root system A_n, we have basis of simple roots

$$\alpha_1 = e_1 - e_2, \alpha_2 = e_3 - e_3, ..., \alpha_n = e_n - e_{n+1},$$

where $e_1, e_2, ..., e_n$ are basis of $\mathbb{Z}^{n+1}$. And we have

$$M = (\sum_{i=1}^{n+1} x_i = 0) \subset \mathbb{Z}^{n+1}, \ N = M^* = \mathbb{Z}^{n+1}/\mathbb{Z} \cdot (1, 1, ..., 1).$$

The fundamental chamber C in $\Sigma_{X(\Phi)}$ corresponds to the basis of simple roots $\alpha_1, \alpha_2, ..., \alpha_n$ via $C = (\alpha_1, \alpha_2, ..., \alpha_n \geq 0) \subset N_{\mathbb{R}}$. C is generated by the dual basis $\alpha_1^*, \alpha_2^*, ..., \alpha_n^*$ of N, where

$$\alpha_1^* = v_1 = (1, 0, 0, ..., 0, 0), \ \alpha_2^* = v_2 = (1, 1, 0, ..., 0, 0), \ ..., \ \alpha_n^* = v_n = (1, 1, 1, ..., 1, 0).$$

So we want to prove the elements of $\tilde{V} = S_{n+1} \cdot \{v_1, v_2, ..., v_n\}$ are vertices of the polytope $\tilde{P}$.

Let $m = \alpha_1 + \alpha_2 + ... + \alpha_n = e_1 - e_{n+1}$, $m \in M$. Then $\langle m, v_i \rangle = 1$ for $i = 1, 2, ..., n$, and $\langle m, v \rangle \leq 1$ for any $v \in \tilde{V}$. The equality holds if and only if $v = (1, x_2, ..., x_n, 0)$ ($x_i = 1$ or 0)

Thus we have a facet F of $\tilde{P}$, $F = \tilde{P} \cap (\langle m, v \rangle = 1) \simeq [0, 1]^{n-1}$.

Thus $v_1, v_2, ..., v_n$ are vertices of the facet F. So $v_1, v_2, ..., v_n$ are vertices of the polytope $\tilde{P}$.

Because $W = W(A_n) = S_{n+1}$ act on $\tilde{P}$ simply transitively on maximal cones $C \in \Sigma_{X(\Phi)}$, if we can prove

$$\tilde{P} = \bigcup_{\sigma \in \Sigma_{X(\Phi)}} Conv(0, v_1, ..., v_n),$$

where $\sigma \in \Sigma_{X(\Phi)}$ is a maximal cone, and $Conv(0, v_1, ..., v_n) = \sigma \cap (m_\sigma \leq 1)$, then the elements of $\tilde{V} = S_{n+1} \cdot \{v_1, v_2, ..., v_n\}$ are vertices of the polytope $\tilde{P}$.

On the one hand, we have

$$\bigcup_{\sigma \in \Sigma_{X(\Phi)}} Conv(0, v_1, ..., v_n) \subset \tilde{P}$$

by the definition of $\tilde{P}$.

On the other hand, since $\tilde{P} \subset (m_\sigma \leq 1)$, we have $\tilde{P} \cap \sigma \subset (m_\sigma \leq 1) \cap \sigma$.

Thus

$$\bigcup_{\sigma \in \Sigma_{X(\Phi)}} (\tilde{P} \cap \sigma) \subset \bigcup_{\sigma \in \Sigma_{X(\Phi)}} ((m_\sigma \leq 1) \cap \sigma).$$

Since the union of the maximal cones is $N_{\mathbb{R}}$, we know

$$\tilde{P} \subset \bigcup_{\sigma \in \Sigma_{X(\Phi)}} ((m_\sigma \leq 1) \cap \sigma) = \bigcup_{\sigma \in \Sigma_{X(\Phi)}} Conv(0, v_1, ..., v_n).$$

Thus we have proved the set of $\tilde{V} = S_{n+1} \cdot \{v_1, v_2, ..., v_n\}$ are vertices of the polytope $\tilde{P}$.

Moreover, we can see then any facet of $\tilde{P}$ can be defined by $\langle m_\sigma, v\rangle = 1$ for some $m \in M$. Actually, we have

$$\tilde{P}^* = \{m \in M_{\mathbb{R}} | \langle m, v\rangle \leq 1, \forall v \in \tilde{P}\}$$
$$= Conv\{e_i - e_j | 1 \leq i, j \leq n+1, i \neq j\} \subset M_{\mathbb{R}}$$

So $\tilde{P}$ is a reflexive polytope.

Next, we prove $\tilde{P}$ is a terminal polytope. We need to show $\tilde{P} \cap N = \{0\} \cup \tilde{V}$.

We have already shown that $\tilde{P} = \bigcup_{\sigma \in \Sigma_{X(\Phi)}} Conv(0, v_1, ..., v_n)$. Since the fan $\Sigma_{X(\Phi)}$ is strictly simplicial, we know

$$Conv(0, v_1, ..., v_n) \cap N = \{0, v_1, ..., v_n\}.$$

So

$$\tilde{P} \cap N = \bigcup_{w \in S_{n+1}} w \cdot Conv(0, v_1, ..., v_n) \cap N$$
$$= \bigcup_{w \in S_{n+1}} w \cdot \{0, v_1, ..., v_n\}$$
$$= \{0\} \cup S_{n+1} \cdot \{v_1, v_2, ..., v_n\}$$
$$= \{0\} \cup \tilde{V}$$

Thus $\tilde{P}$ is a terminal polytope and the corresponding toric variety $\tilde{X}$ has terminal singularities.

◇

Now we can give another proof that P is a terminal polytope when $\Phi = A_n$ by Lemma 4.3.

Lemma 4.4 *Let $\Phi = A_n$ be a root system. Let $P = Conv\{v_1, v_2, ..., v_{3n}\}$ be the polytope corresponding to A_n which is defined in Lemma 2.16. Then P is a terminal polytope. This means the toric variety Z has terminal singularities when $\Phi = A_n$.*

Proof.

Let $\tilde{P} = Conv\{v \mid \rho = \mathbb{R}_{\geq 0} \cdot v$ a ray in $\Sigma_{X(\Phi)}, v \in N$ the primitive generators of $\rho\}$, and $P = Conv\{v_1, v_2, ..., v_{3n}\}$.

Let $\tilde{V} = \{v \mid \rho = \mathbb{R}_{\geq 0} \cdot v$ a ray in $\Sigma_{X(\Phi)}, v \in N$ the primitive generators of $\rho\}$, and $V = \{v_1, v_2, ..., v_{3n}\}$.

We know each v is a vertex of $\tilde{P}$, and $\tilde{P} \cap N = \{0\} \cup \tilde{V}$ by Lemma 4.3.

We want to prove $P \cap N = \{0\} \cup V$.

Obviously we have $P \cap N \supset \{0\} \cup V$, we only need to prove $P \cap N \subset \{0\} \cup V$.

Suppose $\tilde{v} \in P \cap N$, Then $\tilde{v} \in \tilde{P} \cap N = \{0\} \cup \tilde{V}$ since $P \cap N \subset \tilde{P} \cap N$, thus $\tilde{v}$ is a vertex of $\tilde{P}$. Suppose $\tilde{v} \notin V$, then $\tilde{v} \notin P$, a contradiction, so we have $\tilde{v} \in \{0\} \cup V$. Thus $P \cap N = \{0\} \cup V$.

Thus P is a terminal polytope, and the corresponding toric variety Z has terminal singularities.

◇

Consider the two varieties $X(\Phi)$ and $\tilde{X}$ in this chapter when $\Phi = A_n$. Since the fans of these two varieties have the same rays by Lemma 4.3, we have a natural birational map $X(\Phi) \overset{\pi}{\dashrightarrow} \tilde{X}$, which is an isomorphism in codimension one, i.e., no divisors are contracted. Actually, V. Batyrev and M. Blume [BB] showed that π is a morphism.

Lemma 4.5 *[BB] Let $\Phi = A_n$ be a root system. Then the anti-canonical divisor of $X(\Phi)$ is nef and $X(\Phi)$ is an almost Fano variety.*

$-K_{X(\Phi)}$ defines a birational toric morphism $X(\Phi) \xrightarrow{\pi} \tilde{X}$, where $\tilde{X}$ is a Gorenstein toric Fano variety associated with the reflexive polytope $\tilde{P}$.

Question 4.2 is not true in general. Actually, not every v is a vertex of $\tilde{P}$ if we consider the root system D_4. See the following example.

Example 7 *Let $\Phi = D_4$ be the root system, then not every v is a vertex of $\tilde{P}$, and the set of vertices of $\tilde{P}$ is $S_4 \cdot \{(\pm 1, \pm 1, 0, 0)\}$. In particular $\tilde{P}$ has 24 vertices, and $\tilde{P}$ is not a reflexive polytope.*

Proof.

Consider the D_4 root system, the root lattice M is generated by the basis of simple roots $e_1 - e_2, e_2 - e_3, e_3 - e_4$, and $e_3 + e_4$ in the lattice $\mathbb{Z}^4$.

So we have $M = \{x \in \mathbb{Z}^4 | \sum x_i = 0 \bmod 2\}$, and $N = M^* = \mathbb{Z}^4 + \frac{1}{2}(1,1,1,1)$.

The dual basis to the above basis of simple roots is

$$(1,0,0,0), (1,1,0,0), \frac{1}{2}(1,1,1,-1), \frac{1}{2}(1,1,1,1).$$

The set of primitive generators of the rays of the fan of Weyl chambers are obtained from the above dual basis by acting by the Weyl group of D_4, which acts on $\mathbb{Z}^4$ by permutations S_4 and even numbers of sign changes, so $W(D_4) \simeq (\mathbb{Z}/2\mathbb{Z})^3 \rtimes S_4$.

Then the set of generators of rays is given by $(\pm 1, 0, 0, 0), (\pm 1, \pm 1, 0, 0), \frac{1}{2}(\pm 1, \pm 1, \pm 1, \pm 1)$ and the vectors obtained from these by applying permutations.

We have $\frac{1}{2}(1,1,1,1) = \frac{1}{2}(1,1,0,0) + \frac{1}{2}(0,0,1,1)$, i.e., $v' = \frac{1}{2}(1,1,1,1) = \frac{1}{2}v_1 + \frac{1}{2}v_2$, where $v_1 = (1,1,0,0), v_2 = (0,0,1,1)$. So $v' = \frac{1}{2}(1,1,1,1)$ is *not* a vertex of $\tilde{P}$. Actually, we have

$$\tilde{P} = Conv(S_4 \cdot \{(\pm 1, \pm 1, 0, 0)\}),$$

since

$$\frac{1}{2}(\pm 1, \pm 1, \pm 1, \pm 1) = \frac{1}{2}(\pm 1, \pm 1, 0, 0) + \frac{1}{2}(0, 0, \pm 1, \pm 1),$$

and

$$(\pm 1, 0, 0, 0) = \frac{1}{2}(\pm 1, \pm 1, 0, 0) + \frac{1}{2}(\pm 1, \mp 1, 0, 0).$$

We only need to prove each point in $S_4 \cdot \{(\pm 1, \pm 1, 0, 0)\}$ is actually a vertex of $\tilde{P}$. We checked using [PALP] that $\tilde{P} = Conv\{S_4 \cdot \{(\pm 1, \pm 1, 0, 0)\}\}$ has 24 vertices and 33 integral points. So each point in $S_4 \cdot \{(\pm 1, \pm 1, 0, 0)\}$ is actually a vertex of $\tilde{P}$.

Moreover, using [PALP], we can check that $\tilde{P}$ is not a reflexive polytope.

Thus for D_4 root system, not every v_i is a vertex of the polytope $\tilde{P}$, and $\tilde{P}$ is not reflexive.

So $f : X(\Phi) \dashrightarrow \tilde{X}$ has exceptional divisors for D_4, where f is a birational morphism, $\Sigma_{\tilde{X}}$ is the spanning fan of $\tilde{P}$, and $\Sigma_{X(\Phi)}$ is the fan of Weyl Chambers.

$\diamond$

We have already proved that $\tilde{P}$ is reflexive for A_n, but we cannot deduce that P is reflexive for A_n.

Suppose $\tilde{P}$ is a reflexive polytope with vertex set $\tilde{V}$, P is a polytope with vertex set V, and V is a subset of $\tilde{V}$. Then P need not be reflexive.

We have the following example.

Example 8 *$P \subset \tilde{P}$, but $\tilde{P}$ is reflexive and P is not reflexive.*

Proof.

Let P be the polytope with vertices $(0,0,0), (0,0,1), (0,1,0), (1,0,0)$ and $\frac{1}{2}(1,1,1)$, and $\tilde{P}$ be the polytope with vertices $(0,0,0), (0,0,1), (0,1,0), (1,0,0)$, $\frac{1}{2}(1,1,1)$ and $-\frac{1}{2}(1,1,1)$ in the lattice $N = \mathbb{Z}^3 + \mathbb{Z} \cdot \frac{1}{2}(1,1,1)$ with

$$M = N^* = \{(m_1, m_2, m_3) | \sum m_i = 0 \bmod 2\} \subset \mathbb{Z}^3.$$

Then $P \subset \tilde{P}$, but $\tilde{P}$ is reflexive and P is not reflexive.

To prove P is not reflexive, we need to find a face F of P which cannot be defined by $\langle m, v \rangle = 1$ for some $m \in M$. Let F be the face spanned by $\{(0,0,1), (0,1,0), (1,0,0)\}$, defined by $x_1 + x_2 + x_3 = 1$. Then $F = \{v \in N_{\mathbb{R}} | \langle m, v \rangle = 2.\}$, where $m = (2,2,2) \in M$, primitive.

To prove $\tilde{P}$ is a reflexive polytope, we want to show for every face F of $\tilde{P}$, $F = \{v \in N_{\mathbb{R}} | \langle m, v\rangle = 1.\}$ for some $m \in M = \{(m_1, m_2, m_3) | \sum m_i = 0 \text{ mod } 2\}$.

Let F be the face spanned by $\{(0,0,1), (1,0,0), -\frac{1}{2}(1,1,1)\}$, we can compute the normal vector $\vec{n}$ of F.

$$\begin{aligned}\vec{n} &= ((0,0,1) - \frac{1}{2}(1,1,1)) \times ((1,0,0) - \frac{1}{2}(1,1,1)) \\ &= \frac{1}{2}(1,1,3) \times \frac{1}{2}(3,1,1) \qquad = (1,-4,1)\end{aligned}$$

So we have

$$F = \{(x_1, x_2, x_3) | x_1 - 4x_2 + x_3 = 1\},$$

and

$$m = (1,-4,1) \in M = \{(m_1, m_2, m_3) | \sum m_i = 0 \text{ mod } 2\}.$$

Similarly, we can check that for other faces F, $F = \{v \in N_{\mathbb{R}} | \langle m, v\rangle = 1\}$ for some $m \in M$.

Thus $\tilde{P}$ is a reflexive polytope.

◇

Remark 4.6 *The face fan of P defines the the weighted projective space $X = \mathbb{P}(1,1,1,2)$, or equivalently the cone over the Veronese embedding $\mathbb{P}^2 \hookrightarrow \mathbb{P}^5$. The face fan of $\tilde{P}$ defines the cylinder resolution $\tilde{X}$ of X, which is a $\mathbb{P}^1$-bundle over $\mathbb{P}^2$. X and $\tilde{X}$ are Fano varieties, but the anti-canonical divisor $-K_X$ is not Cartier, whereas $-K_{\tilde{X}}$ is Cartier since $\tilde{X}$ is smooth.*

Thus $\tilde{P}$ is reflexive, but P is not reflexive by Lemma 3.20.

Now we have an example of a $(-K)$-flip of a smooth toric variety which is not Gorenstein.

Example 9 *$(-K)$-flip of a smooth toric variety which is not Gorenstein.*

Proof.

Let $N_0 = \mathbb{Z}^4/\mathbb{Z}\cdot(1,-1,1,-1)$, and $\sigma = \langle v_1, v_2, v_3, v_4\rangle_{\mathbb{R}\geq 0}$ where $v_i = \overline{e}_i$. Note $v_1 + v_3 = v_2 + v_4$.

Consider the associated affine toric variety Y. We have $Y \simeq (xy = zt) \subset \mathbb{A}^4_{x,y,z,t}$, where

$$x = \mathcal{X}^{e_1^*+e_3^*},\ y = \mathcal{X}^{e_2^*+e_4^*},\ z = \mathcal{X}^{e_1^*+e_4^*},\ w = \mathcal{X}^{e_2^*+e_3^*}.$$

We have a $\mathbb{Z}/2\mathbb{Z}$ action on Y via $(x,y,z,t) \mapsto (x,-y,-z,t)$. Let X be the quotient. In terms of (N_0,σ), this corresponds to the inclusion $N_0 \subset N = N_0 + \mathbb{Z}\cdot \frac{1}{2}v_4$. Write $v_4' = \frac{1}{2}v_4$.

We consider birational toric morphisms $X_1 \to X$ and $X_2 \to X$ corresponding to the subdivisions Σ_1 and Σ_2 of σ into the cones

$$\{\langle v_1, v_2, v_4'\rangle_{\mathbb{R}\geq 0},\ \langle v_2, v_3, v_4'\rangle_{\mathbb{R}\geq 0}\} \quad \text{and} \quad \{\langle v_1, v_2, v_3\rangle_{\mathbb{R}\geq 0},\ \langle v_1, v_3, v_4'\rangle_{\mathbb{R}\geq 0}\}.$$

Then X_1 is smooth, and X_2 has a unique singularity of type $\frac{1}{2}(1,1,1)$, in particular it is not Gorenstein.

This is because both of $\{v_1, v_2, v_4'\}$ and $\{v_2, v_3, v_4'\}$ are bases of N, whereas $\{v_1, v_2, v_3\}$ is not a basis of N. And we have $X_1 \dashrightarrow X_2$ is a $(-K)$-flip. That is, for $i = 1,2$, the exceptional locus of $X_i \to X$ is a curve $C_i \simeq \mathbb{P}^1$, such that $(-K)\cdot C_1 < 0$ and $(-K)\cdot C_2 > 0$.

This is an example of a $(-K)$-flip of a smooth toric variety which is not Gorenstein.

$\diamond$

CHAPTER 5

RELATION TO MODULI OF POINTS ON $\mathbb{P}^1$ FOR $\Phi = A_n$

5.1 Modular interpretation of $X(A_n)$

In Chapter 3, we give a compactification of the $\mathcal{X}$-cluster variety of finite type for an A, D, E root system up to codimension two. In this chapter, we will use moduli spaces of weighted pointed stable curves to give a nicer compactification of the $\mathcal{X}$-cluster variety of type A root system. This is a toric Gorenstein Fano variety and we can analyze its singularities explicitly.

We consider the smooth projective toric variety $X(A_n)$ given by the fan Σ of Weyl Chambers for the type A_n root system introduced in Chapter 4. Our goal is to relate $X(A_n)$ and the cluster variety for A_n. Fomin and Zelevinsky developed the cluster algebra of finite type associated with A_n. This is the variety corresponding to the cluster algebra.

Let's consider the root system A_n. Let $e_1, e_2, .., e_{n+1}$ be the standard basis of $\mathbb{R}^{n+1}$. Let

$$M_{\mathbb{R}} = \{\sum_{i=1}^{n+1} x_i e_i \in \mathbb{R}^{n+1} \mid \sum_{i=1}^{n+1} x_i = 0\},$$

a subspace of $\mathbb{R}^{n+1}$, then

$$R = \{e_i - e_j \mid i \neq j,\ i, j = 1, ..., n+1\}$$

is a root system of type A_n in $M_{\mathbb{R}}$.

Let $\alpha = e_i - e_j$ be a root, then

$$S_\alpha(x_1, ..., x_{n+1}) = (x_1, ..., x_{j-1}, x_i, x_{j+1}, ..., x_{i-1}, x_j, x_{i+1}, ..., x_{n+1}),$$

so we have $W = W(A_n) = S_{n+1}$.

Let $\alpha_1, ..., \alpha_n$ be the basis of simple roots for A_n given by $\alpha_i = e_i - e_{i+1}, i = 1, .., n$, then $C = \cap_{i=1}^r (\alpha_i \geq 0)$ is a maximal cone in the fan Σ of Weyl chambers of W. It is the fundamental chamber.

V. Batyrev and M. Blume [BB] showed that the toric variety $X(A_n)$ has an interpretation as the Losev-Manin moduli space $\overline{L}_{n+1}$, which are fine moduli spaces of stable $n+3$-pointed chains of projective lines, where the stability condition is such that some of the points are allowed to coincide.

B. Hassett introduced moduli spaces of weighted pointed stable curves, and showed that $\overline{L}_{n+1}$ is a particular case of these spaces with weights $\mathcal{A}_n = (1, (\frac{1}{n+1})^{n+1}, 1)$ [BH]. Here by $(1, (\frac{1}{n+1})^{n+1}, 1)$ we mean $(1, \frac{1}{n+1}, ..., \frac{1}{n+1}, 1)$. There are $n+1$ occurrences of the weight $\frac{1}{n+1}$.

Thus $X(A_n)$ is isomorphic to the moduli space of weighted pointed stable curves $\overline{M}_{0,\mathcal{A}_n}$ with weights $\mathcal{A}_n = (1, (\frac{1}{n+1})^{n+1}, 1)$.

B. Hassett also showed that $\mathbb{P}^n$ is isomorphic to the moduli space of weighted pointed stable curves $\overline{M}_{0,\mathcal{A}}$ with weights $\mathcal{A} = ((\frac{1}{n+1})^{n+2}, 1)$, which is the Kapranov model of $\overline{M}_{0,n}$. Under this model, $\mathbb{P}^n$ is the Hassett moduli space where the marked point x_{n+3} is distinct from x_i for any i, and $x_1, ..., x_{n+1}, x_{n+2}$ do not all coincide.

Now consider two GIT quotients introduced by Mumford. [MF]

Let $(\mathbb{P}^1)^{n+3}//_\alpha SL(2)$ and $G(2, n+3)//_\alpha H$ be two generic GIT quotients, where $H = (\mathbb{C}^\times)^{n+3}/\mathbb{C}^\times$. They depend on α, which is given by a choice of an ample line bundle on $(\mathbb{P}^1)^{n+3}$ in the first case and a linearization of the Plucker line bundle in the second. See [Kap] fore more details. In fact, there is a chamber decomposition of the ample cone such that the quotient only depends on the face of the chamber decomposition containing α. (This is the theory of Variation of GIT quotient, see e.g. [DH].) We will assume that α is generic in the sense that it lies in the interior of a chamber.

M.M.Kapranov proved that these two GIT quotients are isomorphic under the so-called Gelfand-MacPherson correspondence[[Kap], Theorem 2.4.7]. Let $Q_\alpha = (\mathbb{P}^1)^{n+3}//_\alpha SL(2) = G(2, n+3)//_\alpha H$.

GIT quotients have weights $(a_1, a_2, ..., a_n)$ such that $\sum_{i=1}^n a_j = 2$, and Hassett moduli spaces have weights $(a_1, a_2, ..., a_n)$ such that $\sum_{i=1}^n a_j > 2$. B. Hassett showed that GIT quotients of $(\mathbb{P}^1)^n$ can be interpreted as "small-parameter limits" of the moduli spaces $\overline{M}_{0,\mathcal{A}}$ as $\sum_{i=1}^n a_j \to 2$, where $\mathcal{A} = (a_1, ..., a_n)$ is the weight.

Theorem 5.1 ([BH], Theorem 8.2) *Let $\mathcal{D}_n$ be the domain of weight data*

$$\mathcal{D}_n = \{(a_1, ..., a_n) \in \mathbb{R}^n \mid 0 < a_i \leq 1 \quad \textit{and} \quad a_1 + a_2 + \cdots + a_n > 2.\}$$

Let $\mathcal{T} = (t_1, ..., t_n)$ be a typical *linearization, i.e., $t_1 + \cdots + t_n = 2$ and $t_{i_1} + \cdots + t_{i_r} \neq 1$ for any $\{i_1, ..., i_r\} \subset \{1, ..., n\}$. Let $\mathcal{Q}(\mathcal{T})$ be the GIT quotient of $(\mathbb{P}^1)^n$ under this typical linearization. Let*

$$\begin{aligned} U = &\{(u_1, ..., u_n) \in \mathbb{Q}^n : 0 < u_i < 1 \quad \textit{and} \\ &u_{i_1} + ... + u_{i_r} \neq 1 \quad \textit{for any} \quad \{i_1, ..., i_r\} \subset \{1, ..., n\}\}. \end{aligned}$$

Then $U \cap \mathcal{D}_n$ is contained in an open chamber of $\mathcal{D}_n$. For each set of weight data $\mathcal{A} \in U \cap \mathcal{D}_n$, there is a natural isomorphism

$$\overline{M}_{0,\mathcal{A}} \xrightarrow{\simeq} \mathcal{Q}(\mathcal{T}).$$

We can use Theorem 5.1 to prove that we can choose α so that $Q_\alpha = (\mathbb{P}^1)^{n+3}//_\alpha SL(2)$ is isomorphic to the moduli space of weighted pointed stable curves $\overline{M}_{0,\mathcal{A}}$ with some weights.

Lemma 5.2 *Let $a = \frac{1}{n+1} + \epsilon, 0 < \epsilon \ll 1$, we can choose an ϵ so that*

$$\alpha = (a^{n+2}, 2 - (n+2)a)$$

is a typical linearization. Then $Q_\alpha = (\mathbb{P}^1)^{n+3}//_\alpha SL(2)$ is isomorphic to the moduli space of weighted pointed stable curves $M_{0,\mathcal{W}}$ with weights $\mathcal{W} = (a^{n+2}, 1)$.

Proof.

Let $a = \frac{1}{n+1} + \epsilon, 0 < \epsilon \ll 1$, we can choose an ϵ so that

$$\alpha = (a^{n+2}, 2 - (n+2)a)$$

is a typical linearization, and Q_α is "generic", i.e., α lies in the interior of a chamber. So we get a line bundle on $(\mathbb{P}^1)^{n+3}$, and it induces a GIT quotient

$$Q_\alpha = (\mathbb{P}^1)^{n+3}//_\alpha SL(2).$$

If we choose a rational ϵ, then for weight $\mathcal{W} = (a^{n+2}, 1)$, we can check that $\mathcal{W} \in U \cap \mathcal{D}_n$, which satisfies the condition of Theorem 5.1. So we get a natural isomorphism

$$\overline{M}_{0,\mathcal{W}} \simeq \mathcal{Q}(\alpha) = Q_\alpha = (\mathbb{P}^1)^{n+3}//_\alpha SL(2).$$

Thus $Q_\alpha = (\mathbb{P}^1)^{n+3}//_\alpha SL(2)$ is isomorphic to the moduli space of weighted pointed stable curves $M_{0,\mathcal{W}}$ with weights $\mathcal{W} = (a^{n+2}, 1)$.

$\diamond$

Let $\overline{M}_{0,\mathcal{B}}$ be the moduli space of weighted pointed stable curves with weights $\mathcal{B} = (1, a^{n+1}, 1)$. Then using the theory of moduli space of weighted pointed stable curves, we can see $\overline{M}_{0,\mathcal{B}}$ can be interpreted as the blowup of identity in $X(A_n)$.

Similarly, let $Q = Q_\alpha$ defined in Lemma 5.2, then Q can be interpreted as the blow up of $n+2$ points in general position in $\mathbb{P}^n$.

Lemma 5.3 *Let $\overline{M}_{0,\mathcal{B}}$ be the moduli space of weighted pointed stable curves with weights $\mathcal{B} = (1, a^{n+1}, 1)$. Let $\tilde{X}(A_n) \to X(A_n)$ be blowup of $e \in T \subset X(A_n)$. Then $\overline{M}_{0,\mathcal{B}}$ is isomorphic to $\tilde{X}(A_n)$.*

Similarly, let $Q = Q_\alpha = (\mathbb{P}^1)^{n+3}//_\alpha SL(2)$ defined in Lemma 5.2. Then Q is isomorphic to the blow up of $n+2$ points in general position in $\mathbb{P}^n$.

Proof.

B. Hassett showed that $X(A_n)$ is isomorphic to the moduli space of weighted pointed stable curves $\overline{M}_{0,\mathcal{A}_n}$ with weights $\mathcal{A}_n = (1, (\frac{1}{n+1})^{n+1}, 1)$.

Under this identification, we consider an open subset which consists of curves, $(C, p_1, p_2, ..., p_{n+3})$, where $C \simeq \mathbb{P}^1$, p_1 and p_{n+3} are disjoint from other points with weight 1.

We may assume $\varphi : (C, p_1, p_{n+3}) \overset{\sim}{\longleftarrow} (\mathbb{P}^1, 0, \infty)$. There is a choice here: $\varphi \rightsquigarrow \varphi \circ \lambda$

$\lambda \in \mathbb{C}^\times$, $\lambda : \mathbb{P}^1 \longrightarrow \mathbb{P}^1, z \longmapsto \lambda \cdot z$

$\mathbb{C}^\times = Aut(\mathbb{P}^1, 0, \infty) \subset PGL(2) = Aut(\mathbb{P}^1)$ acts on $\mathbb{P}^1$.

So we get an open subset $U \simeq (\mathbb{C}^\times)^{n+1}/\mathbb{C}^\times$.

The identity $e \in T = (\mathbb{C}^\times)^{n+1}/\mathbb{C}^\times$, corresponds to

$$p_1 = 0, p_{n+3} = \infty, p_2 = p_3 = \cdots = p_{n+2}.$$

Under the modular interpretation, we have a reduction morphism

$$f : \overline{M}_{0,\mathcal{A}_n} \longrightarrow \overline{M}_{0,\mathcal{B}}.$$

We have $\overline{M}_{0,\mathcal{A}_n} \setminus p \simeq \overline{M}_{0,\mathcal{B}} \setminus E$ under the modular interpretation.

We consider the subset E, which consists of curves $(C, p_1, p_2, ..., p_{n+3})$, where $C \simeq \mathbb{P}^1$, p_1 and p_{n+3} are disjoint from other points with weight 1. And $p_2, ..., p_{n+2} \in \mathbb{P}^1 \setminus \{\infty\} = \mathbb{A}^1$, they have weights a and at most n points coincide. We will prove $E \simeq \mathbb{P}^{n-1}$:

Since $Aut(\mathbb{A}^1) = (z \longmapsto az + b)$, so we can translate, such that $\sum_{i=2}^{n+2} p_i = 0$. Then because they do not all coincide, we have $(p_2, ..., p_{n+2}) \neq (0, ..., 0)$, and have a residual $\mathbb{C}^\times$ action on $\mathbb{A}^1$, $z \longmapsto az$.

So we get

$$E \simeq \frac{\{(x_1, x_2, ..., x_{n+1}) | \sum x_i = 0\} \setminus \{0\}}{\mathbb{C}^\times} = \mathbb{P}^{n-1}$$

Now we have

$$X(A_n) \setminus e \simeq \overline{M}_{0,\mathcal{B}} \setminus E$$

under the modular interpretation.

Since $\overline{M}_{0,\mathcal{B}} \longrightarrow X(A_n)$ is birational morphism, and $\overline{M}_{0,\mathcal{B}}$, $X(A_n)$ are both smooth and have dimension n. We know $\overline{M}_{0,\mathcal{B}}$ is isomorphic to the blowup of e in $X(A_n)$, i.e., $\overline{M}_{0,\mathcal{B}} \simeq \tilde{X}(A_n)$.

Similarly, we can prove that Q is isomorphic to the blow up of $n+2$ points in general position in $\mathbb{P}^n$. That's exactly the first step of the alternate approach to Kapranov's moduli space in [[BH], 6.2]. $\diamond$

Now by the following lemma in Hassett's paper. We have birational morphisms between $X(A_n)$, $\tilde{X}(A_n)$, Q and $\mathbb{P}^n$.

Lemma 5.4 *(Reduction Morphism) ([BH], Theorem 4.1) Let* $\mathcal{A} = (a_1, ..., a_n)$ *and* $\mathcal{B} = (b_1, ..., b_n)$ *be collections of weight data so that* $b_j \leq a_j$ *for each* $j = 1, ..., n$. *Let* $\overline{M}_{0,\mathcal{A}}$ *and* $\overline{M}_{0,\mathcal{B}}$ *be the moduli space of weighted pointed stable curves with weights*

$\mathcal{A}$ *and* $\mathcal{B}$ *respectively. Then there exists a natural birational reduction morphism*

$$\rho_{\mathcal{B},\mathcal{A}} : \overline{M}_{0,\mathcal{A}} \longrightarrow \overline{M}_{0,\mathcal{B}}.$$

Given an element $(C, s_1, ..., s_n) \in \overline{M}_{0,\mathcal{A}}$ *,* $\rho_{\mathcal{B},\mathcal{A}}(C, s_1, ..., s_n)$ *is obtained by successively collapsing components of* C *along which* $K_C + b_1 s_1 + ... + b_n s_n$ *fails to be ample.*

Theorem 5.5 *For* $X(A_n)$, $\tilde{X}(A_n)$, Q, $\mathbb{P}^n$ *defined above, we have the following birational morphisms*

$$\begin{array}{ccccc} \tilde{X}(A_n) & \xrightarrow{\pi} & Q & \xrightarrow{Bl^{n+2}} & \mathbb{P}^n \\ & \searrow^{Bl_e} & & \nearrow_{\eta} & \\ & & X(A_n) & & \end{array}$$

Proof.

We have shown that $X(A_n)$, $\tilde{X}(A_n)$, Q, $\mathbb{P}^n$ are moduli spaces of weighted pointed stable curves with weights $(1, (\frac{1}{n+1})^{n+1}, 1), (1, a^{n+1}, 1), (a^{n+2}, 1), ((\frac{1}{n+1})^{n+2}, 1)$, $a = \frac{1}{n+1} + \epsilon, 0 < \epsilon \ll 1$ for proper ϵ.

Thus by Lemma 5.4, we have the following diagram of birational morphisms:

$$\begin{array}{ccccc} \tilde{X}(A_n) & \xrightarrow{\pi} & Q & \xrightarrow{Bl^{n+2}} & \mathbb{P}^n \\ & \searrow^{Bl_e} & & \nearrow_{\eta} & \\ & & X(A_n) & & \end{array}$$

$\diamond$

Proposition 5.6 *We have the following maps*

$$\begin{array}{ccccccc} X(A_n) & \xleftarrow{Bl_e} & \tilde{X}(A_n) & \xrightarrow{\pi} & Q & \xrightarrow{Bl^{n+2}} & \mathbb{P}^n \\ & & & & & & \\ D & \longleftarrow & \tilde{D} & \xrightarrow{\pi} & D_Q & \longrightarrow & \bar{D} \end{array}$$

where D *is a divisor of* $X(A_n)$, $\tilde{D} = D' \cup E$, $\bar{D}$ *is a choice of toric boundary divisor for* $\mathbb{P}^n$, $D_Q = \bar{D}' \cup E_p \cup E_q$. E *is the exceptional divisor over* $e \in X(A_n)$, E_p *and* E_q *are the divisors over two points* $p, q \in \mathbb{P}^n$.

Then $D, \tilde{D}, D_Q$ and $\bar{D}$ have $2n, 2n+1, n+3$ and $n+1$ components respectively, and $\mathcal{X} = Q \setminus D_Q$ is the cluster variety for $\Phi = A_n$.

Proof.

Suppose $C = \cap_{i=1}^r (\alpha_i \geq 0)$ is the fundamental chamber in the fan Σ, and $C = \langle \omega_1, ..., \omega_n \rangle \subset N_{\mathbb{R}}$, which corresponds to a 0-stratum $p \in X(A_n)$. Let $D_{\omega_1}, ..., D_{\omega_n}$ be the toric boundary divisors corresponding to the generators of C, equivalently, the boundary divisors containing p.

Let $D = \sum_{i=1}^n D_{\omega_i} + \sum_{i=1}^n \overline{(\mathcal{X}^{\alpha^i} = 1)}$, $\tilde{D} = D' \cup E$, where E is the exceptional divisor of the identity $e \in X(A_n)$.

We have $K_{X(A_n)} + D \neq 0$, $K_{\tilde{X}(A_n)} + \tilde{D} \neq 0$, but $K_Q + D_Q = 0$, $K_{\mathbb{P}^n} + \bar{D} = 0$.

Since Q is isomorphic to the blowup of $n+2$ points of $\mathbb{P}^n$ in general position. By changing coordinates, we may assume we blow up these $n+2$ points $p = p_1 = (1, 0, ..., 0), p_2 = (0, 1, ..., 0), ..., p_{n+1} = (0, 0, ..., 1)$ and $q = p_{n+2} = (1, 1, ..., 1)$. Then $S_{n+1} = W(A_n)$ act on Q transitively. In fact $W = S_{n+1}$ acts on all the four spaces. But Q is not toric and not Fano, and number of components of D_Q is $(n+1) + 2 = n+3$.

$\bar{D}$ is the toric boundary divisor of $\mathbb{P}^n$. What is $\bar{D} \subset \mathbb{P}^n$? (in terms of Kapranov's description)

Let's see Kapranov model of $\mathbb{P}^n$: It is a Hassett space with weights $((\frac{1}{n+1})^{n+2}, 1)$. We may assume $x_{n+3} = \infty$, $x_1, x_2, ..., x_{n+2} \in \mathbb{A}^1 = \mathbb{C}$, no $n+1$ points can coincide. then

$$\mathbb{P}^n \simeq \frac{\mathbb{C}^{n+2} \setminus \{(x, x, ..., x) | x \in \mathbb{C}\}}{Aut\mathbb{C}} = \frac{\{(x_1, x_2, ..., x_{n+2}) | \sum x_i = 0\} \setminus \{0\}}{\mathbb{C}^\times}.$$

Now we may assume

$$\bar{D} = \bar{D}_1 + \bar{D}_2 + \cdots + \bar{D}_{n+1} = \{x_1 = x_2\} + \{x_2 = x_3\} + \cdots + \{x_{n+1} = x_{n+2}\}.$$

We can check these hyperplanes in $\mathbb{P}^n \simeq \frac{\{(x_1,x_2,\ldots,x_{n+2}) | \sum x_i = 0\} \setminus \{0\}}{\mathbb{C}^\times}$ are in general position. Then $D_Q = \bar{D}' \cup E_p \cup E_q$.

With this choice of p_i, let $\overline{Q}$ be the blowup of $n+1$ coordinate points $p_1, \ldots, p_{n+1}$ in $\mathbb{P}^n$, then the map $Q \longrightarrow \overline{Q}$ is blowup of p_{n+2}, i.e., the identity element.

Now $D_Q = \bar{D}' \cup E_{p_1} \cup E_{p_{n+2}}$, $D_{\overline{Q}} = \bar{D}' \cup E_{p_1}$. In particular, $D_{\overline{Q}}$ has $n+2$ components and D_Q has $n+3$ components. $\mathcal{X} = Q \setminus D_Q = \overline{Q} \setminus D_{\overline{Q}}$ is the $\mathcal{X}$ cluster variety for $\Phi = A_n$. $\overline{Q}$ is toric, but $D_{\overline{Q}}$ is not the toric boundary.

So we have the following diagram:

$$\begin{array}{cccccc} \text{not toric} & \tilde{X}(A_n) & \xrightarrow{\pi} & Q & & \\ & \downarrow{\scriptstyle Bl_e} & & \downarrow{\scriptstyle Bl_e} & \searrow{\scriptstyle Bl^{n+2}} & \\ \text{toric} & X(A_n) & \xrightarrow{\overline{\pi}} & \overline{Q} & \xrightarrow{Bl^{n+1}} & \mathbb{P}^n \end{array}$$

$\overline{Q}$ is toric with $2(n+1)$ toric boundary divisors and $D_{\overline{Q}}$ have $n+2$ components.

We can verify that this description coincides with the description of cluster variety in [GHKII].

So we have $\mathcal{X} = Q \setminus D_Q = \overline{Q} \setminus D_{\overline{Q}}$ is the $\mathcal{X}$ cluster variety for $\Phi = A_n$.

$\diamond$

5.2 Compactification of A_n cluster variety

Let $\overline{Q}$ be the blowup of $n+1$ coordinate points $(1,0,\ldots,0)$,$(0,1,\ldots,0)$,...,$(0,0,\ldots,1)$ in $\mathbb{P}^n$.

We know the fan of $\mathbb{P}^n$ are maximal cones $\langle e_0, \ldots, \hat{e}_i, \ldots, e_n \rangle_{\mathbb{R}_{\geq 0}}$ in $\mathbb{R}^{n+1}/\mathbb{R} = \overline{N}_{\mathbb{R}}$. Thus the fan of $\overline{Q}$ are maximal cones $\langle e_0, .., \hat{e}_i, \ldots, \hat{e}_j, \ldots, e_n, \sum_{k \neq i} e_k \rangle_{\mathbb{R}_{\geq 0}}$.

Let Σ' be the fan of the toric variety $\overline{Q}$, $\Sigma' \subset \overline{N}_{\mathbb{R}} = \mathbb{R}^{n+1}/\mathbb{R} \cdot (1, \ldots, 1)$. The rays of the fan are

$$\rho_i = \mathbb{R}_{\geq 0} \cdot f_i, i = 0, 1, ..., n$$

$$\rho_{n+i} = \mathbb{R}_{\geq 0} \cdot -f_i = \mathbb{R}_{\geq 0}(f_0 + f_1 + ... + f_i + ... + f_n), i = 0, 1, ..., n$$

Lemma 5.7 *Let $\bar{P}' = Convex(f_i, -f_i, i = 0, 1, ..., n) \subset N_{\mathbb{R}} = \mathbb{R}^{n+1}/\mathbb{R} \cdot (1, ..., 1)$, then each f_i and $-f_i$ is a vertex of the polytope $\bar{P}'$. Moreover, we have $\bar{P}'$ is a reflexive polytope.*

Proof.

See appendix.

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Let the fan $\Sigma_{\overline{Q}'}$ be the cone over all the faces of $\bar{P}'$. It has the same rays ρ_i, ρ_{n+i} as Σ'. Let $\overline{Q}'$ be the toric variety corresponds to the fan $\Sigma_{\overline{Q}'}$.

Then there is a birational map $\overline{Q} \dashrightarrow \overline{Q}'$, which is an isomorphism up to codimension two since they have the same rays.

Lemma 5.8 *$\overline{Q}'$ is a Gorenstein toric fano variety.*

Proof.

It follows from Lemma 5.7.

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Theorem 5.9 *For $n \geq 2$, we have the following maps:*

$$X(A_n) \xrightarrow{\overline{\pi}} \overline{Q} \overset{\pi'}{\dashrightarrow} \overline{Q}',$$

such that $\overline{Q}'$ is a Gorenstein toric Fano variety, and π' is a birational map, which is an isomorphism up to codimension two.

When n is even, $\overline{Q}'$ is smooth. When $n = 2k-1$ is odd, it has $\binom{2k}{k}$ terminal singularities of type Cone($\mathbb{P}^{k-1} \times \mathbb{P}^{k-1}$), and $\overline{Q}' = (\mathbb{P}^1)^{n+1}//\mathbb{C}^\times$ with weights $(1,1,...1,-1,-1,...,-1)$.

Proof.

Let

$$\bar{P}' = Convex(f_i, -f_i, i = 0, 1, ..., n) \subset \overline{N}_{\mathbb{R}} = \mathbb{R}^{n+1}/\mathbb{R} \cdot (1, ..., 1),$$

which satisfy $\sum_{i=1}^{n} f_i = 0$, and let

$$P' = Convex(f_i, -f_i, i = 0, 1, ..., n) \subset N_{\mathbb{R}} = \mathbb{R}^{n+1},$$

$$R = P'^* = \{\mu \in M_{\mathbb{R}} | \langle \mu, v_i \rangle \geq -1, \forall i\} = (|x_i| \leq 1) \subset M_{\mathbb{R}} = \mathbb{R}^{n+1}$$

and $\overline{R} = \bar{P}'^*$

Then $\overline{R} = R \cap H \subset H \subset \mathbb{R}^{n+1}$, where $H = \{\sum x_i = 0 | x_i \in \mathbb{R}\}$ is a hyperplane.

Let W and $\overline{Q}'$ be the toric variety correspond to P' and $\bar{P}'$ respectively. We know $W = (\mathbb{P}^1)^{n+1}$.

We have the $\mathbb{C}^\times \subset T = N \otimes \mathbb{C}^\times = (\mathbb{C}^\times)^{n+1}$ acting on $W = (\mathbb{P}^1)^{n+1}$ by the diagonal map $\lambda \longmapsto (\lambda, \lambda, ..., \lambda)$. So by the GIT construction, we have $\overline{Q}' = W//\mathbb{C}^\times = (\mathbb{P}^1)^{n+1}//\mathbb{C}^\times$ as a GIT quotient.

$$\overline{Q}' = W//\mathbb{C}^\times = \begin{cases} W^s/\mathbb{C}^\times, & \text{if} \quad n \text{ is even} \\ W^{ss}//\mathbb{C}^\times, & \text{if} \quad n \text{ is odd} \end{cases}$$

where $W^s \subset W$ are the stable points in W, and $W^{ss} \subset W$ are the semistable points in W.

When n is even, H does not contain vertices of R, then $W^s = W^{ss}$.

Thus $\overline{Q}' = W//\mathbb{C}^\times = W^s/\mathbb{C}^\times$ is a geometric quotient. There are no strictly semistable points, no finite stabilizers, and the action of $\mathbb{C}^\times$ on W^s is free.

In particular, $\overline{Q}'$ is a smooth Fano variety if n is even, so then $\overline{Q}'$ is terminal.

When n is odd, $\overline{Q}' = W//\mathbb{C}^\times = W^{ss}//\mathbb{C}^\times$ is a categorical quotient. And the action of $\mathbb{C}^\times$ on W^{ss} has some fixed points.

For n odd, there are some singularities corresponding to those fixed points of the action of $\mathbb{C}^\times$ on W^{ss}. Let's analyze these singularities.

Let $w = (\pm 1, \pm 1, ..., \pm 1)$ be a fixed point under the action of $\mathbb{C}^\times$ in W. So there are $\frac{n+1}{2}$ 1's and $\frac{n+1}{2}$ -1's. It corresponds to $\frac{n+1}{2}$ points equal to 0 and $\frac{n+1}{2}$ points equal to ∞ in the setting of the moduli space of $n+1$ pointed curves.

We can analyze locally at the point w, we have the local action of $\mathbb{C}^\times$ on $(\mathbb{A}^1)^{n+1}$ with weights $w = (\pm 1, \pm 1, ..., \pm 1)$.

Without loss of generality, we can assume $w = (1, 1, ..., 1, -1, -1, ..., -1)$. Let $k = \frac{n+1}{2}$, then we have k 1's and k -1's.

For any vertex w of $\overline{R}$, we have a corresponding point $\bar{p} \in \overline{Q}'$. Let $p \in W^{ss}$ be a point in the inverse image of $\bar{p}$. Then we have the local model $(p \in W^{ss})/\mathbb{C}^\times = (\bar{p} \in W//\mathbb{C}^\times)$, where $(p \in W^{ss})/\mathbb{C}^\times = Spec(\mathbb{C}[\sigma^* \cap M]^G)$, and $\mathbb{C}[\sigma^* \cap M] = k[x_1, x_2, ..., x_{2k}], G = \mathbb{C}^\times$, thus

$$\begin{aligned} Spec(k[x_1, x_2, ..., x_{2k}]^{\mathbb{C}^\times}) &= k[\mathbb{P}^{k-1} \times \mathbb{P}^{k-1} \underset{Segre}{\hookrightarrow} \mathbb{P}^{k^2-1}] \\ &= k[x_i x_j | 1 \leq i \leq k, k+1 \leq j \leq 2k] \subset k[x_1, x_2, ..., x_{2k}] \end{aligned}$$

Next, we prove these singularities of type Cone($\mathbb{P}^{k-1} \times \mathbb{P}^{k-1}$) are terminal singularities.

We have the Segre embedding, $\mathbb{P}^{k-1} \times \mathbb{P}^{k-1} \overset{S}{\hookrightarrow} \mathbb{P}^{k^2-1}$, where the map S is given by $\mathcal{O}(1) \boxtimes \mathcal{O}(1) = (1, 1)$.

Let $L = \mathcal{O}(1) \boxtimes \mathcal{O}(1)$, a very ample line bundle over $\mathbb{P}^{k-1} \times \mathbb{P}^{k-1}$, and let L^* be the dual bundle of L. Then we have the cylinder resolution of singularities of the cone over $\mathbb{P}^{k-1} \times \mathbb{P}^{k-1}$:

$$\pi : L^* = \tilde{X} \longrightarrow X = Cone(\mathbb{P}^{k-1} \times \mathbb{P}^{k-1}),$$

where $E \subset L^*$ is the exceptional divisor over the singular point $p \in X$, and E is a copy of $\mathbb{P}^{k-1} \times \mathbb{P}^{k-1}$.

Suppose we have $K_{\tilde{X}} = \pi^* K_X + a \cdot E$. We want to show $a > 0$, so then $p \in X$ is a terminal singularity. We have $E|_E = L^* = (-1, -1)$.

By the adjunction formula, we know

$$K_E = (K_{\tilde{X}} + E)|_E = (\pi^* K_X + (a+1)E)|_E = (a+1) \cdot E|_E = (a+1) \cdot (-1, -1).$$

Since E is a copy of $\mathbb{P}^{k-1} \times \mathbb{P}^{k-1}$, we have $K_E = (-k, -k)$.

Thus $(a+1) \cdot (-1, -1) = (-k, -k)$, and $a = k - 1$.

Since $k \geq 2$, so we have $a > 0$, and these singularities of type Cone($\mathbb{P}^{k-1} \times \mathbb{P}^{k-1}$) are terminal singularities.

And for any point $w = (\pm 1, \pm 1, ..., \pm 1) \in R$, k 1's and k -1's. we have one terminal singularity. Thus, $\overline{Q}'$ has $\binom{2k}{k}$ terminal singularities, each of type Cone($\mathbb{P}^{k-1} \times \mathbb{P}^{k-1}$).

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Example 10 *When $n = 3$, $(\mathbb{P}^1)^{n+1} // \mathbb{C}^\times$ is a Fano 3-fold with 6 ordinary double points. Number of singular points:* $\binom{2k}{k} = \binom{4}{2} = 6$, $k = \frac{n+1}{2} = 2$

Now we have the compactification of the $\mathcal{X}$ cluster variety for $\Phi = A_n$.

Theorem 5.10 *Let $\mathcal{X}$ be the $\mathcal{X}$-cluster variety for the root system A_n. Then there exists a toric terminal Gorenstein Fano variety $\overline{Q}'$ and a divisor $\bar{D}'$ with $n+2$ components, such that $\mathcal{X}$ is isomorphic to $\overline{Q}' \backslash \bar{D}'$ up to codimension two. Moreover, we have $K_{\overline{Q}'} + \bar{D}' \sim 0$ and $(\overline{Q}', \bar{D}')$ is log canonical.*

Proof. By Lemma 5.8 we know $\overline{Q}'$ is a toric terminal Gorenstein Fano variety.

We have the map

$(Q, D) \xrightarrow{\pi} (\overline{Q}, \bar{D}) \overset{\pi'}{\dashrightarrow} (\overline{Q}', \bar{D}')$

where π is the blowup of identity $e \in \overline{Q}$, and π' is a birational map, which is an isomorphism up to codimension two.

Because $K_Q + D \sim 0$ and $\pi_* K_Q = K_{\overline{Q}}, \pi_* D = \bar{D}$, we have

$$K_{\overline{Q}} + \bar{D} = \pi_*(K_Q + D) \sim \pi_*(0) = 0.$$

Since π' is an isomorphism up to codimension two, it follows that

$$K_{\overline{Q}'} + \bar{D}' \sim 0.$$

For (Q, D), it is the blowup of $\mathbb{P}^n$. So Q is smooth and we can check that D is a normal crossing divisor. Then it is divisorial log terminal and also log canonical.

Then $(\overline{Q}, \bar{D})$ and $(\overline{Q}', \bar{D}')$ are both log canonical by Lemma 3.8 in Chapter 3.

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APPENDIX A

PROOF OF LEMMA

Lemma A.1 *Let* $\rho_i = \mathbb{R}_{\geq 0} \cdot v_i$*, where* $v_i \in N$ *is a primitive generator of* N*,* $v_i = f_i$*,* $v_{n+i} = f_i - \sum_{k \neq i} |b_{ik}| f_k$*,* $v_{2n+i} = -f_i$*,* $i = 1, ..., n$*, and* $f_1, f_2, ..., f_n$ *is the standard basis of* $\mathbb{Z}^n$*. Then* $P := Conv(v_1, ...v_{3n}) \subset N_{\mathbb{R}}$ *is a convex polytope containing* 0 *in its interior and with vertices* $\{v_1, ...v_{3n}\}$*.*

Proof.

First, we show $P := Conv(v_1, ...v_{3n}) \subset N_{\mathbb{R}}$ contains 0 in its interior.

We need to prove that there exist λ_i such that

$$\sum_{i=1}^{3n} \lambda_i v_i = 0, \quad 0 < \lambda_i < 1 \quad \forall i, \quad \sum_{i=1}^{3n} \lambda_i = 1.$$

Take $\lambda'_i = 1$ for $n+1 \leq i \leq 2n$, and $\lambda'_i = 2$ for $2n+1 \leq i \leq 3n$, then

$$\begin{aligned} \sum_{i=n+1}^{3n} \lambda'_i v_i &= \sum_{i=1}^{n} (f_i - \sum_{k \neq i} |b_{ik}| f_k - 2f_i) \\ &= \sum_{i=1}^{n} (-f_i - \sum_{k \neq i} |b_{ik}| f_k) \\ &= -\sum_{i=1}^{n} \lambda'_i f_i \end{aligned}$$

for some $\lambda'_i > 0$, $1 \leq i \leq n$.

Then we have shown that there exists $\lambda'_i > 0$, $1 \leq i \leq 3n$, such that $\sum_{i=1}^{3n} \lambda'_i v_i = 0$.

So then

$$\frac{\sum_{i=1}^{3n}\lambda'_i v_i}{\sum_{i=1}^{3n}\lambda'_i} = \sum_{i=1}^{3n}\frac{\lambda'_i}{\sum_{j=1}^{3n}\lambda'_j}v_i = 0, \quad \lambda'_i > 0, \forall i.$$

Let $\lambda_i = \frac{\lambda'_i}{\sum_{j=1}^{3n}\lambda'_j}$, then $\sum_{i=1}^{3n}\lambda_i = 1$, and $\sum_{i=1}^{3n}\lambda_i v_i = 0$, $0 < \lambda_i < 1$.

This means $P := Conv(v_1, ...v_{3n}) \subset N_{\mathbb{R}}$ contains 0 in its interior.

Second, we show that $v_1, ..., v_{3n}$ are vertices of P.

Suppose v_i is not a vertex of P, and $1 \leq i \leq n$ then $v_i = \sum\limits_{j\neq i}\mu_j v_j$ for $\sum \mu_j = 1, \mu_j \geq 0$

Since $v_j = f_j$, $v_{n+j} = f_j - \sum_{k=1,k\neq j}^{n}|b_{jk}|\cdot f_k, i = 1, ..., n$, then $f_i = \sum\limits_{j\neq i}\mu_j f_j + \sum_{j=1}^{n}\mu_{n+j}(f_j - \sum_{k=1,k\neq j}^{n}|b_{jk}|\cdot f_k)$

For simply laced root systems, $|b_{jk}| = 0$ or 1, so $f_i = \mu_{n+i}f_i$, thus $\mu_{n+i} = 1$ and $\mu_j = 0$ if $j \neq n+i$. Then we have $v_i = v_{n+i}$, a contradicition.

If $n+1 \leq i \leq 2n$, we can similarly get a contradiction.

Thus $v_1, ...v_{2n}$ are all vertices of P.

Next, we prove that v_i is also a vertex of P for $2n+1 \leq i \leq 3n$.

Suppose v_{2n+i} is not a vertex of P for $1 \leq i \leq n$, then we have

$v_{2n+i} = \sum\limits_{j\neq i}\mu_j v_j$ for $\sum \mu_j = 1, \mu_j \geq 0$, $1 \leq j \leq 3n$, $j \neq 2n+i$.

Since $v_j = f_j$, $v_{n+j} = f_j - \sum_{k=1,k\neq j}^{n}|b_{jk}|\cdot, v_{2n+j} = -f_j$, $j = 1, ..., n$, then $-f_i = \sum_{j=1}^{n}\mu_j f_j + \sum_{j=1}^{n}\mu_{n+j}(f_j - \sum_{k=1,k\neq j}^{n}|b_{jk}|\cdot f_k) - \sum\limits_{j\neq i}\mu_{2n+j}f_j$

Thus we have $1 = \sum\limits_{i,j \text{ an edge}}\mu_{n+j}$.

So $\mu_j = 0$, $1 \leq j \leq n$, or $2n+1 \leq j \leq 3n$, and $\mu_{n+j} = 0$, if ij is not an edge.

But then $v_{2n+i} = \sum\limits_{i,j \text{ an edge}}\mu_{n+j}(f_j - \sum_{k=1,k\neq j}^{n}|b_{jk}|f_k)$, a contradiction.

Thus we have shown that $v_1, ...v_{3n}$ are all vertices of P.

$\diamond$

Lemma A.2 *Let $P := Conv(v_1, ..., v_{3n}) \subset N_{\mathbb{R}}$ be the convex polytope defined in Lemma 2.16. Then 0 is a unique lattice point contained in the interior of P,* *moreover,* $P \cap N = \{0, v_1, ..., v_{3n}\}$*, i.e.,* P *is terminal.*

Proof. We need to show that 0 is a unique lattice point contained in the interior of P and the boundary lattice points of P are precisely the vertices of the polytope.

For the sake of contradiction, suppose $m \in P \cap N$, $m \neq 0$, Let $m = \sum_{j=1}^{n} a_j f_j$, a_j are integers. Then

$$m = \sum_{j=1}^{n} a_j f_j = \sum_{j=1}^{3n} \lambda_j v_j, \qquad 0 \leq \lambda_j < 1, \quad \sum_{j=1}^{3n} \lambda_j = 1.$$

So we have

$$m = \sum_{j=1}^{n} a_j f_j = \sum_{j=1}^{3n} \lambda_j v_j = \sum_{j=1}^{n} \lambda_j f_j + \sum_{j=1}^{n} \lambda_{n+j}(f_j - \sum_{k=1,k\neq j}^{n} |b_{jk}| f_k) - \sum_{j=1}^{n} \lambda_{2n+j} f_j.$$

Comparing the coefficients of f_i in both side, we see

$$a_i = \lambda_i + \lambda_{n+i} - \lambda_{2n+i} - \sum_{k=1,k\neq j}^{n} \lambda_{n+j} |b_{ij}|$$

Since $m \neq 0$, suppose there exist i such that $a_i > 0$, since $\sum_{j=1}^{3n} \lambda_j = 1$ and a_i is an integer, we have $a_i = 1$. Thus

$$\lambda_i + \lambda_{n+i} = \lambda_{2n+i} + \sum_{k=1,k\neq j}^{n} \lambda_{n+j} |b_{ij}| + a_i \geq 1.$$

Then we have $\lambda_i + \lambda_{n+i} = 1$, and $\lambda_j = 0$ if $j \neq i$, $j \neq n+i$, so

$$m = \sum_{j=1}^{n} a_j f_j = \lambda_i f_i + \lambda_{n+i}(f_i - \sum_{k=1,k\neq i}^{n} |b_{ik}| f_k) = f_i - \lambda_{n+i} \sum_{j=1,j\neq i}^{n} |b_{ik}| f_k$$

So λ_{n+i} is an ineger, and $0 \leq \lambda_j < 1$, $\lambda_{n+i} = 0$.

Now we have $m = f_i$, a contradiction. So it is not an interior point.

If $a_j \leq 0$ for all j, and there exists i such that $a_i < 0$, since $\sum_{j=1}^{3n} \lambda_j = 1$ and a_i is an integer, we have $a_i = -1$, thus

$$\lambda_{2n+i} + \sum_{k=1,k\neq j}^{n} \lambda_{n+j} |b_{ij}| = \lambda_i + \lambda_{n+i} + 1 \geq 1.$$

So $\lambda_{2n+i} + \sum_{k=1,k\neq j}^{n} \lambda_{n+j}|b_{ij}| = 1$

For simply laced root systems, we have

$$|b_{ij}| = \begin{cases} 1, & \text{if} \quad \text{i j an edge} \\ 0, & \text{if} \quad \text{i j is not an edge} \end{cases}$$

Then for any i, there are at most 3 j's such that $|b_{ij}| = 1$. If there are 1 or 2 j's such that $|b_{ij}| = 1$, then by a similar proof, we can see it is impossible. If there are 3 j's such that $|b_{ij}| = 1$, that's the case in D_n root system. Then $i = n-2$, we have $b_{n-2,n-3} = 1, b_{n-2,n-1} = 1, b_{n-2,n} = 1$, and $b_{ij} = 0$ otherwise.

Then we have $\lambda_{3n-2} + \lambda_{2n-3} + \lambda_{2n-1} + \lambda_{2n} = 1$, and $\lambda_j = 0$ otherwise. Thus

$$\begin{aligned} m &= \sum_{j=1}^{n} a_j f_j \\ &= -\lambda_{3n-2} f_{n-2} + \lambda_{2n-3}(f_{n-3} - f_{n-2} - f_{n-4}) + \lambda_{2n-1}(f_{n-1} - f_{n-2}) + \lambda_{2n}(f_n - f_{n-2}) \\ &= -\lambda_{3n-2} f_{n-2} - \lambda_{2n-3} f_{n-2} - \lambda_{2n-3} f_{n-4} - \lambda_{2n-3} f_{n-2} - \lambda_{2n-3} f_{n-2} \\ &+ \lambda_{2n-3} f_{n-3} + \lambda_{2n-1} f_{n-1} + \lambda_{2n} f_n \end{aligned}$$

Because $a_j \leq 0$ for all j, so $\lambda_{2n-3} = \lambda_{2n-1} = \lambda_{2n} = 0$ and $\lambda_{3n-2} = 1$.

Thus $m = -f_{n-2}$, a contradiction, this is not an interior point. So we have already prove that 0 is a unique lattice point contained in the interior of P and the boundary lattice points of P are precisely the vertices of the polytope.

Thus $P \cap N = \{0, v_1, ..., v_{3n}\}$, P is terminal.

$\diamond$

Lemma A.3 *Let $\bar{P}' = Convex(f_i, -f_i, i = 0, 1, ..., n) \subset N_{\mathbb{R}} = \mathbb{R}^{n+1}/\mathbb{R} \cdot (1, ..., 1)$, then each f_i and $-f_i$ is a vertex of the polytope $\bar{P}'$. Moreover, we have $\bar{P}'$ is a reflexive polytope.*

Proof. Suppose f_0 is not a vertex of the polytope $\bar{P}'$, then

$$f_0 = \sum_{j=1}^{n} a_j f_j + \sum_{k=0}^{n} b_j(-f_j)$$

for $a_j, b_j \geq 0$ and $\sum_{j=1}^{n} a_j + \sum_{j=0}^{n} b_j = 1$.

Thus we have $(-1 - b_0)f_0 + (a_1 - b_1)f_1 + ... + (a_n - b_n)f_n = 0$.

But in $N_{\mathbb{R}} = \mathbb{R}^{n+1}/\mathbb{R} \cdot (1, ..., 1)$, we have $f_0 + f_1 + ... + f_n = 0$, i.e., $\lambda(f_0 + f_1 + ... + f_n) = 0$. Then we know $(-1 - b_0)f_0 + (a_1 - b_1)f_1 + ... + (a_n - b_n)f_n = \lambda(f_0 + f_1 + ... + f_n)$. Since $f_1, f_2, ..., f_n$ are basis of $\mathbb{R}^n$, we have $\lambda = -1 - b_0 \neq 0 \leq -1$ and $\lambda = a_j - b_j \geq -1$.

Thus $b_0 = 0$, $a_j = 0\ \forall j \geq 0$, and $b_j = 1\ \forall j \geq 1$.

That's a contradiction.

This means that f_0 is actually a vertex of the polytope $\bar{P}'$.

Similarly $-f_0$ is also a vertex of the polytope $\bar{P}'$.

Next, we prove that $\mu = \pm f_i\ i = 1, ..., n$ is also a vertex of the polytope $\bar{P}'$.

Suppose $f_i = \sum_{j=0}^{n} a_j f_j + \sum_{k=0}^{n} b_j(-f_j)$, for $a_j, b_j \geq 0$ and $\sum_{j=0}^{n} a_j + \sum_{j=0}^{n} b_j = 1$.

Since $f_0 = -\sum_{j=0}^{n} f_j$, then

$$f_i = \sum_{j=0}^{n}(a_j - b_j)f_j = \sum_{j=1}^{n}[(a_j - b_j) - (a_0 - b_0)]f_j = \sum_{j=1}^{n}(v_j - v_0)f_j.$$

Thus $v_i - v_0 = 1$ and $v_j - v_0 = 0$ for $j \neq i$.

So $a_i = 1$, $a_j = 0$ for $j \neq i$, and $b_j = 0$ for any j. It means that $f_i = f_i$.

Thus $\mu = \pm f_i\ i = 1, ..., n$ are vertex of the polytope $\bar{P}'$.

So we have shown that each f_i and $-f_i$ is a vertex of the polytope $\bar{P}'$.

Now, we prove that $\bar{P}'$ is a reflexive polytope.

Let

$$P' = Convex(f_i, -f_i, i = 0, 1, ..., n) \subset N_{\mathbb{R}} = \mathbb{R}^{n+1},$$

$$R = P'^* = \{\mu \in M_{\mathbb{R}} | \langle \mu, v_i \rangle \geq -1, \forall i\} = (|x_i| \leq 1) \subset M_{\mathbb{R}} = \mathbb{R}^{n+1}$$

and $\overline{R} = \bar{P}'^*$ be the dual polytope of $\bar{P}'$.

Then $\overline{R} = R \cap H \subset H \subset \mathbb{R}^{n+1}$, where $H = \{\sum x_i = 0 | x_i \in \mathbb{R}\}$ is a hyperplane.

Let $P = Convex(f_i, -f_i, i = 0, 1, ..., n) \subset N_{\mathbb{R}} = \mathbb{R}^{n+1}$, and $Q = P^* = \{\mu \in M_{\mathbb{R}} | \langle \mu, v_i \rangle \geq -1, \forall i\} = (|x_i| \leq 1) \subset M_{\mathbb{R}} = \mathbb{R}^{n+1}$.

Let's compute its vertices of $\overline{R}$.

The vertices of Q are $(\pm 1, \pm 1, ..., \pm 1)$, here we have $n+1$ of ± 1's.

Then the edge of the hyperplane will be

$$\{(\pm 1, \pm 1, ..., x, ..., \pm 1) | x \in [-1, 1]\}.$$

If H intersect the edge at some point, then $m + x = 0$, this require $m = \pm 1, 0$, since $x \in [-1, 1]$.

Suppose $n+1$ is even, we have m odd. Thus $m = \pm 1$, $x = \mp 1$.

So the vertices of $\overline{R}$ are $(\pm 1, ..., \pm 1)$, where the number of $+$'s is equal to the number of $-$'s.

Suppose $n+1$ is odd, we have m even. Thus $m = 0$, $x = 0$.

So the vertices are $(\pm 1, \pm 1, ..., 0, ..., \pm 1, \pm 1)$, where the number of $+$'s is equal to the number of $-$'s.

In both cases, the dual polytope $\overline{R}$ has integer vetices,

So $\bar{P}'$ and $\overline{R}$ are both reflexive polytope.

$\diamond$

APPENDIX B

PARI/GP PROGRAM TO COMPUTE AMPLE DIVISORS

```
/*-----------------------------How to use?-----------------------------------*/


\\Example: D4 case

\\(0). Make sure the file "xie" does not exist.

If it exist, rename it or delete it.

\\(1). Type "L=Start(4, [ [1,-1,0,0], [0,-1,1,0], [0,-1,0,1], [-1,1,-1,-1] ])",

where "4" stands for the dimension.

\\(2). Type "Search(L)", this will create the file "xie" with useful information.

\\(3). Type "Check()", this will return "1" if everything is good,

otherwise return "0" and display the "bad" pair of cones.

\\(4). If one wants to read details of the file "xie", one can type "L[i]",

where "i" is the order of the cone in L.

\\      For instance, in the above example, one will see that the command

"Check()" will display "[4,28,3]".

Then if one wants to see the generators of the pair of cone,
```

```
one can type "L[4]" and "L[28]".

/*------Function part, don't use it, jump to the main part in the following------*
Int2mat(Integer, Len)=
{    my(V, n);
    n=Integer;
    V=vector(Len, i, 0);
    for(i=1, Len,
        if(n%2==1, V[i]=1, V[i]=-1);
        n=(n-n%2)/2;
    );
    V=Vecrev(V);
    return(V);  }
/*This part will creat the initial matrices, input the dimension "Dim",
it returns the matrices to start at*/
Create(Dim)=
{    my(MM);
    MM=vector(2^Dim, i, matdiagonal(Int2mat(i-1, Dim)));
    MM=Vecrev(MM);
    return(List(MM));}

/*Given "Mat" and "Vec" corresponding to the cone and vector respectively,
determine whether the vector is in the cone by returning 1 or 0*/
VecIn(MAT, VEC)=
{    my(Sol,ok);
    Sol=matsolve(MAT, VEC~);
```

```
    ok=1;
    for(i=1, length(Sol), if(Sol[i]<0, ok=0; break));
    return(ok);}

/*In this part, given "MAT" and "VEC" and an inDteger "c",
it will return the matrix which is the matrix MAT
with the c-th column replace by VEC*/
Replace(MAT,VEC,c)=
{    my(NMAT);
    NMAT=MAT;
    NMAT[,c]=VEC~;
    return(NMAT); }

/*"LIST" is a list of matrices, this part will find the cone which contains
"VEC", and replace it by the new ones*/
Renew(LIST,VEC)=
{    my(DEL, L, M);
    L=LIST;
    DEL=List([]);
    for(i=1,length(LIST),
        if(VecIn(L[i], VEC)==1, listput(DEL, i));   );
    DEL=Vecrev(DEL);
    for(j=1, length(DEL),
        M=L[DEL[j]];
        listpop(L, DEL[j] );
        for(k=1, length(VEC),
```

```
            if(matdet(Replace(M, VEC, k))!=0, listput(L, Replace(M, VEC, k)));
        );
    );
    return(L);}

/*Given two matrices, "M1,M2", this part will determine whether they are
adjacent and if YES, give the sum of coefficient*/
Comb(M1,M2)=
{    my(num, M,LV,v1,v2,ok,u1,u2,b1,b2, S,Ker,c);
    ok=1;
    LV=List([]);
    for(i=1,matsize(M1)[1],
        v1=M1[,i];
        for(j=1, matsize(M2)[1],
            if(v1==M2[,j], listput(LV, v1));
        );
    );

    if(  length(LV)!=matsize(M1)[1]-1,
        ok=0,
        for(i=1,matsize(M1)[1],
            v1=M1[,i];
            v2=M2[,i];
            u1=u2=1;
            for(j=1, length(LV), if(v1==LV[j], u1=0); if(v2==LV[j], u2=0));
            if(u1==1, b1=v1);
```

```
            if(u2==1, b2=v2);         );


        listput(LV, b1+b2);

        M=matrix(matsize(M1)[1],matsize(M1)[1], a,b,LV[b][a]);

        Ker=matker(M)[,1]~;

        S=sum(x=1, length(LV)-1, Ker[x]);

        c=-Ker[length(LV)];

        S=S/c;

        );

    if(ok==1, return(S),return("null"));}


/*---- Main part, to use this code, start with the following commands-----*/


/*Given the dimension "Dim", this part will first initialize all the cones by

using the code "Create(Dim)", and we are given the list "LLIST"*/

/*Given the list "LIST" of matrices, and a vector of vector, say "W"

(pay attention to the order of vectors in W),

this part will Renew the LIST with respect to the vectors in W in the right order.

Start(Dim, W)=

{    my(LLIST);

    LLIST=Create(Dim);

    for(i=1, #W, LLIST=Renew(LLIST, W[i]));

    return(LLIST);}



/*Write the results about L to file named "xie"*/
```

```
Search(LIST)=
{ my();
for(i=1,length(LIST),
for(j=i,length(LIST),
write("xie", [i,j,Comb(LIST[i],LIST[j])]);
);
);
    write("xie", LIST); }


/*Read the file "xie", and check if any sum is >2,
if not, return 1, otherwise, reuten 0*/


Check()=
{ my(VV,ok);
VV=readvec("xie");
    L=VV[#VV];
ok=1;
for(i=1, length(VV)-1,
if(VV[i][3]!="null",
if(VV[i][3]>2,
                ok=0;
                print(VV[i]);
            );
);
);
return(ok);}
```

```
/*Type is 1 or 2, corresponding to A or D repectively*/

Run(Type, Dim)=

{    my(VV, L,a);

        if(Type ==1,

        VV=vector(Dim, i, vector(Dim, j, if(j==i,1, j==i+1, -1, j==i-1, -1, 0))),

        Type ==2,

        VV=vector(Dim, i, vector(Dim, j, if(j==i,1, j==i+1, -1, j==i-1, -1, 0)));

        VV[Dim-2][Dim]=-1;

        VV[Dim-1][Dim]=0;

        VV[Dim][Dim-2]=-1;

        VV[Dim][Dim-1]=0;

    );


    L=Start(Dim, VV);

    Search(L);

    a=Check();

    return(a);

}
```

For example, if run this program, and then type “Run(1,4)”, this correspond to root system A_4, it will input

“[10, 17, 3]”

This means $-K$ is not nef and there exists only one toric 1-statum C, which corresponding to the cone L[10] and L[17], such that $-K \cdot C = -1$.

If we type “Run(2,4)”, this correspond to root system D_4, it will input [4, 14, 3] [6, 16, 3] [7, 17, 3] [8, 15, 3]

This means $-K$ is not nef and there exists four toric 1-statum C, such that $-K \cdot C = -1$.

If we type “Run(2,5)”, this correspond to root system D_5, it will input

[9, 33, 3] [11, 35, 3] [12, 36, 3] [13, 34, 3] [14, 30, 3] [16, 29, 4] [17, 32, 3] [19, 31, 3]

This means $-K$ is not nef and there are seven toric 1-stratum C such that $(-K) \cdot C = -1$ and one toric 1-stratum C such that $(-K) \cdot C = -2$.

```
sage: points = [(1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1), (-1,0,0,0), (0,-1,0,
....: 0), (0,0,-1,0), (0,0,0,-1),(1,-1,0,0), (-1,1,-1,0), (0,-1,1,-1), (0,0,-1,1
....: )]
sage: p = LatticePolytope(points)
[sage:
[sage: q = p.polar()
[sage: p.nvertices()
[12
sage: p.npoints()
[13
sage: q.nvertices()
[23
sage: q.npoints()
[46
sage: q.vertices()
[
N(-1, -1,  1,  1),
N(-1, -1,  1,  0),
N( 1, -1, -1,  1),
N( 1,  1,  1,  0),
N( 1,  0,  0, -1),
N( 1,  1,  0, -1),
N( 1,  1,  1,  1),
N( 1,  1, -1, -1),
N( 1, -1, -1, -1),
N( 0, -1,  0, -1),
N( 0,  1, -1, -1),
N( 0,  1,  1,  1),
N( 0,  1,  0, -1),
N( 0,  1,  1,  0),
N(-1,  0,  1,  0),
N(-1,  0,  0, -1),
N(-1,  0,  1,  1),
N(-1,  0,  0,  1),
N(-1,  0, -1, -1),
N(-1, -1, -1,  1),
N(-1, -1, -1, -1),
N(-1, -1,  0, -1),
N(-1,  0, -1,  0)
in 4-d lattice N
sage: 
```

Figure 1. lattice points of dual polytope of A4

```
sage: points = [(1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1), (-1,0,0,0), (0,-1,0,
....: 0), (0,0,-1,0), (0,0,0,-1),(1,-1,0,0), (-1,1,-1,-1), (0,-1,1,0), (0,-1,0,1
....: )]
sage: p = LatticePolytope(points)
sage: q = p.polar()
sage: p.nvertices()
12
sage: p.npoints()
13
sage: q.nvertices()
24
sage: q.npoints()
47
sage: q.vertices()

N(-1,  0, -1, -1),
N(-1,  0, -1,  1),
N(-1,  0,  1, -1),
N( 1,  1,  0,  0),
N( 1,  1,  1,  0),
N( 1,  0,  1, -1),
N( 1,  1,  0,  1),
N( 1, -1,  0, -1),
N( 1,  0, -1, -1),
N( 0, -1,  1, -1),
N( 1, -1, -1, -1),
N( 1, -1, -1,  0),
N( 0, -1, -1,  1),
N( 0,  1,  0,  1),
N( 0,  1,  1,  1),
N( 0,  1,  0,  0),
N( 0,  1,  1,  0),
N( 1,  0, -1,  1),
N(-1,  0,  1,  1),
N(-1, -1,  0,  1),
N(-1, -1, -1,  1),
N(-1, -1, -1, -1),
N(-1, -1,  1, -1),
N(-1, -1,  1,  0)
in 4-d lattice N
sage:
```

Figure 2. lattice points of dual polytope of D4

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